# *Элементы теории поля*

$u = u(\mathbf{r}) = u(x, y, z)$ — скалярное поле. ( $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k},\ \ r = \sqrt{x^2 + y^2 + z^2}$ )

$\operatorname{grad} u = \dfrac{\partial u}{\partial x}\mathbf{i} + \dfrac{\partial u}{\partial y}\mathbf{j} + \dfrac{\partial u}{\partial z}\mathbf{k}$ — градиент скалярного поля $u$.

$(\operatorname{grad} u, \mathbf{l})$ — производная поля $u$ по направлению $\mathbf{l}$.

$\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ — векторное поле

$\operatorname{div}\mathbf{a} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$ — дивергенция векторного поля $\mathbf{a}$.

$\operatorname{rot}\mathbf{a} = \left(\dfrac{\partial a_z}{\partial y} - \dfrac{\partial a_y}{\partial z}\right)\mathbf{i} + \left(\dfrac{\partial a_x}{\partial z} - \dfrac{\partial a_z}{\partial x}\right)\mathbf{j} + \left(\dfrac{\partial a_y}{\partial x} - \dfrac{\partial a_x}{\partial y}\right)\mathbf{k}$ — ротор векторного поля $\mathbf{a}$.

$$\operatorname{rot}\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$\nabla = \mathbf{i}\dfrac{\partial}{\partial x} + \mathbf{j}\dfrac{\partial}{\partial y} + \mathbf{k}\dfrac{\partial}{\partial z}$ — символический вектор набла.

$$\operatorname{grad} u = \nabla u, \qquad \operatorname{div}\mathbf{a} = (\nabla, \mathbf{a}), \quad \operatorname{rot}\mathbf{a} = \nabla \times \mathbf{a}.$$

Пример1. $\operatorname{grad} r = \dfrac{\mathbf{r}}{r}$

Пример 2. $\operatorname{grad} f(r) = f'(r)\dfrac{\mathbf{r}}{r}$

Пример 3. $\operatorname{grad}(\mathbf{a}, \mathbf{r}) = \mathbf{a}$

Пример 4. (№4412) $\operatorname{grad}|\mathbf{c} \times \mathbf{r}|^2 = 2\mathbf{r}(\mathbf{c}, \mathbf{c}) - 2\mathbf{c}(\mathbf{c}, \mathbf{r})$

Пример 5. $\operatorname{rot}\operatorname{grad} u = \mathbf{0}, \qquad \operatorname{div}\operatorname{grad} u = \Delta u$

Пример 6. $\operatorname{div}\mathbf{r} = \mathbf{3}$

$$\operatorname{div} r^5\mathbf{r} = \left(\nabla, r^5\mathbf{r}\right) =$$

$$= \left(\nabla, r^5 \overset{\downarrow}{\mathbf{r}}\right) + \left(\nabla, \overset{\downarrow}{r^5} \mathbf{r}\right) = r^5\left(\nabla, \mathbf{r}\right) + \left(\nabla r^5, \mathbf{r}\right) =$$

$$= r^5 \operatorname{div}\mathbf{r} + \left(\operatorname{grad} r^5, \mathbf{r}\right) = 3r^5 + \left(5r^4\frac{\mathbf{r}}{r}, \mathbf{r}\right) = 8r^5$$

Пример 7 $\operatorname{rot}\mathbf{r} = \mathbf{0}$

Пример 8. $\operatorname{rot}\mathbf{a}\times\mathbf{r} = 2\mathbf{a}$

$$\mathbf{a}\times\left(\mathbf{b}\times\mathbf{c}\right) = \mathbf{b}\left(\mathbf{a},\mathbf{c}\right) - \mathbf{c}\left(\mathbf{a},\mathbf{b}\right)$$

$$\operatorname{rot}\mathbf{a}\times\mathbf{r} = \nabla\times\left(\mathbf{a}\times\mathbf{r}\right) = \mathbf{a}\left(\nabla,\mathbf{r}\right) - \left(\mathbf{a},\nabla\right)\mathbf{r} = 3\mathbf{a} - \mathbf{a} = \mathbf{2a}$$

$$\left(\left(\mathbf{a},\nabla\right) = a_x\frac{\partial}{\partial x} + a_y\frac{\partial}{\partial y} + a_z\frac{\partial}{\partial z},\ \frac{\partial\mathbf{r}}{\partial x} = \mathbf{i}, \frac{\partial\mathbf{r}}{\partial y} = \mathbf{j}, \frac{\partial\mathbf{r}}{\partial z} = \mathbf{k}\right)$$

Пример 9. (№4437 а)) $\operatorname{rot} f\left(r\right)\mathbf{c} = \frac{f'\left(r\right)}{r}\mathbf{r}\times\mathbf{c}$

$$\operatorname{rot} f\left(r\right)\mathbf{c} = \nabla\times f\left(r\right)\mathbf{c} = \nabla f\left(r\right)\times\mathbf{c} = \operatorname{grad} f\left(r\right)\times\mathbf{c} = \frac{f'\left(r\right)}{r}\mathbf{r}\times\mathbf{c}$$

пример 10 (№4437 б)) $\operatorname{rot}\mathbf{c}\times f\left(r\right)\mathbf{r} = 2f\left(r\right)\mathbf{c} + \frac{f'\left(r\right)}{r}\left(\mathbf{c}\left(\mathbf{r},\mathbf{r}\right) - \mathbf{r}\left(\mathbf{c},\mathbf{r}\right)\right)$

$$\operatorname{rot}\mathbf{c}\times f\left(r\right)\mathbf{r} = \nabla\times\left(\mathbf{c}\times f\left(r\right)\mathbf{r}\right) = \mathbf{c}\left(\nabla, f\left(r\right)\mathbf{r}\right) - \left(\mathbf{c},\nabla\right) f\left(r\right)\mathbf{r} =$$

$$= \mathbf{c}\left(3f\left(r\right) + \frac{f'\left(r\right)}{r}\left(\mathbf{r},\mathbf{r}\right)\right) - \left(\mathbf{c}f\left(r\right) + \left(\mathbf{c}, \frac{f'\left(r\right)}{r}\mathbf{r}\right)\mathbf{r}\right) =$$

$$= 2f\left(r\right)\mathbf{c} + \frac{f'\left(r\right)}{r}\left(\mathbf{c}\left(\mathbf{r},\mathbf{r}\right) - \mathbf{r}\left(\mathbf{c},\mathbf{r}\right)\right)$$

Домашнее задание.

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