

ЗАНЯТИЕ 1 КРИВОЛИНЕЙНЫЕ ИНТЕГРАЛЫ

4221

$$\int_{\Gamma} (x+y) dl$$

$$O(0,0), A(1,0), B(0,1)$$

$$\int_{OA} (x+y) dl = \int_0^1 x dx = \frac{1}{2}, \int_{AB} (x+y) dl = \int_{AB} dl = \sqrt{2},$$

Ответ: $\sqrt{2} + 1$.

4222

$$I = \int_{\Gamma} y^2 dl; x = a(t - \sin t), y = a(1 - \cos t);$$

$$dl = a\sqrt{(1 - \cos t)^2 + \sin^2 t} dt = a\sqrt{2 - 2\cos t} dt = 2a \sin \frac{t}{2} dt$$

$$I = \int_0^{2\pi} a^2 (1 - \cos t)^2 2a \sin \frac{t}{2} dt = 8a^3 \int_0^{2\pi} \sin^5 \frac{t}{2} dt = 16a^3 \int_0^{\pi} \sin^5 u du = 32a^3 \frac{4!!}{5!!} = \frac{256}{15} a^3$$

4225

$$I = \int_{\Gamma} (x^{4/3} + y^{4/3}) dl, \Gamma: x^{2/3} + y^{2/3} = a^{2/3};$$

$$x = a \cos^3 t, y = a \sin^3 t;$$

$$dl = a\sqrt{(-3\cos^2 t \sin t)^2 + (-3\sin t \cos^2 t)^2} dt = 3a \cos t \sin t dt, 0 \leq t \leq \pi/2;$$

$$I = 12a^{7/3} \int_0^{\pi/2} (\cos^4 t + \sin^4 t) \cos t \sin t dt = 24a^{7/3} \int_0^{\pi/2} \cos t \sin^5 t dt = 24a^{7/3} \cdot \frac{1}{6} = 4a^{7/3}$$

4229

$$I = \int_{\Gamma} \sqrt{x^2 + y^2} dl, \Gamma: x^2 + y^2 = ax$$

$$x = \frac{a}{2}(1 + \cos t), y = \frac{a}{2} \sin t; dl = \frac{a}{2} dt;$$

$$I = \frac{a^2}{2\sqrt{2}} \int_0^{2\pi} \sqrt{1 + \cos t} dt = \frac{a^2}{\sqrt{2}} \int_0^{\pi} \sqrt{1 + \cos t} dt = a^2 \int_0^{\pi} \cos \frac{t}{2} dt = 2a^2$$

4232

$$x = e^{-t} \cos t, y = e^{-t} \sin t, z = e^{-t};$$

$$dl = \sqrt{(e^{-t}(-\cos t - \sin t))^2 + (e^{-t}(-\sin t + \cos t))^2 + e^{-2t}} dt = e^{-t} \sqrt{3}$$

$$L = \sqrt{3} \int_0^{+\infty} e^{-t} dt = \sqrt{3}$$

4234

$$(x-y)^2 = a(x+y), \quad x^2 - y^2 = \frac{9}{8}z^2;$$

$$O(0,0,0), \quad A(x_0, y_0, z_0)$$

$$x = \frac{1}{\sqrt{2}}(u-v), \quad y = \frac{1}{\sqrt{2}}(u+v),$$

$$2v^2 = \sqrt{2}au, \quad -2uv = \frac{9}{8}z^2;$$

$$u = \frac{\sqrt{2}v^2}{a}, \quad uv = -\frac{9}{16}z^2, \quad \frac{\sqrt{2}v^3}{a} = -\frac{9}{16}z^2,$$

$$v^3 = -\frac{9a}{16\sqrt{2}}z^2;$$

$$v = -\left(\frac{9a}{16\sqrt{2}}\right)^{1/3} z^{2/3} = -\frac{3^{2/3}a^{1/3}}{2^{3/2}} z^{2/3}, \quad u = \frac{\sqrt{2}}{a^{1/3}} \left(\frac{9}{16\sqrt{2}}\right)^{2/3} z^{4/3} = \frac{3^{4/3}}{a^{1/3}2^{5/2}} z^{4/3};$$

$$dl = \sqrt{1 + \left(-\frac{3^{2/3}a^{1/3}}{2^{3/2}} \frac{2}{3} z^{-1/3}\right)^2 + \left(\frac{3^{4/3}}{a^{1/3}2^{5/2}} \frac{4}{3} z^{1/3}\right)^2} dz = \sqrt{1 + \left(-\frac{a^{1/3}}{2^{1/2}3^{1/3}} z^{-1/3}\right)^2 + \left(\frac{3^{1/3}}{a^{1/3}2^{1/2}} z^{1/3}\right)^2} dz =$$

$$= \left(\frac{a^{1/3}}{2^{1/2}3^{1/3}} z^{-1/3} + \frac{3^{1/3}}{a^{1/3}2^{1/2}} z^{1/3}\right) dz;$$

$$L = \int_0^{z_0} \left(\frac{a^{1/3}}{2^{1/2}3^{1/3}} z^{-1/3} + \frac{3^{1/3}}{a^{1/3}2^{1/2}} z^{1/3}\right) dz = \frac{3^{2/3}a^{1/3}}{2^{3/2}3^{1/3}} z_0^{2/3} + \frac{3^{4/3}}{a^{1/3}2^{5/2}} z_0^{4/3}$$

4238

$$\int_{\Gamma} x^2 dl, \quad \Gamma: x^2 + y^2 + z^2 = a^2, \quad x + y + z = 0$$

$$\int_{\Gamma} x^2 dl = \frac{1}{3} \int_{\Gamma} (x^2 + y^2 + z^2) dl = \frac{a^2}{3} \int_{\Gamma} dl = \frac{2\pi a^2}{3}$$

$$z = -(x + y)$$

$$x^2 + y^2 + (x + y)^2 = a^2$$

$$x^2 + xy + y^2 = \frac{a^2}{2}$$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(u - v) \\ y = \frac{1}{\sqrt{2}}(u + v) \end{cases}$$

$$u^2 + v^2 + \frac{1}{2}(u^2 - v^2) = \frac{a^2}{2}$$

$$3u^2 + v^2 = a^2$$

$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x + y + z = 0 \end{cases} \begin{cases} u = \frac{a}{\sqrt{3}} \cos t \\ v = a \sin t \end{cases} \begin{cases} x = \frac{a}{\sqrt{2}} \left(\frac{1}{\sqrt{3}} \cos t - \sin t \right) = \frac{\sqrt{2}a}{\sqrt{3}} \cos \left(t + \frac{\pi}{3} \right) \\ y = \frac{a}{\sqrt{2}} \left(\frac{1}{\sqrt{3}} \cos t + \sin t \right) = \frac{\sqrt{2}a}{\sqrt{3}} \cos \left(t - \frac{\pi}{3} \right) \\ z = -\frac{\sqrt{2}a}{\sqrt{3}} \cos t \end{cases}$$

$$dl = a \sqrt{\frac{1}{2} \left(-\frac{1}{\sqrt{3}} \sin t - \cos t \right)^2 + \frac{1}{2} \left(-\frac{1}{\sqrt{3}} \sin t + \cos t \right)^2 + \frac{2}{3} \sin^2 t} dt = a dl$$

$$I = \int_0^{2\pi} \frac{2a^3}{3} \cos^2 \left(t + \frac{\pi}{3} \right) dt = \frac{2\pi a^3}{3}$$

4240

$$\int_{\Gamma} z dl, \quad \Gamma: x^2 + y^2 = z^2, y^2 = ax \quad a, x, y, z > 0$$

$$x^2 + y^2 = z^2, y^2 = ax$$

$$x = \frac{y^2}{a},$$

$$\frac{y^4}{a^2} + y^2 = z^2, \quad z = \frac{\sqrt{y^4 + a^2 y^2}}{a}$$

$$\begin{aligned}
dl &= \sqrt{\left(\frac{2y}{a}\right)^2 + 1 + \left(\frac{2y^3 + a^2y}{a\sqrt{y^4 + a^2y^2}}\right)^2} dy = \sqrt{\left(\frac{2y}{a}\right)^2 + 1 + \left(\frac{2y^2 + a^2}{a\sqrt{y^2 + a^2}}\right)^2} dy = \\
&= \sqrt{\frac{4y^2}{a^2} + 1 + \frac{4y^4 + 4a^2y^2 + a^4}{a^2(y^2 + a^2)}} dy = \sqrt{\frac{8y^4 + 9a^2y^2 + 2a^4}{a^2(y^2 + a^2)}} dy \\
\int_{\Gamma} z dl &= \int_0^a \frac{\sqrt{y^4 + a^2y^2}}{a} \sqrt{\frac{8y^4 + 9a^2y^2 + 2a^4}{a^2(y^2 + a^2)}} dy = \frac{1}{a^2} \int_0^a y \sqrt{8y^4 + 9a^2y^2 + 2a^4} dy = \\
&= \frac{1}{2a^2} \int_0^{a^2} \sqrt{8u^2 + 9a^2u + 2a^4} du = \frac{a^2}{2} \int_0^1 \sqrt{8v^2 + 9v + 2} dv = a^2 \sqrt{2} \int_0^1 \sqrt{v^2 + \frac{9}{8}v + \frac{1}{4}} dv = \\
&= a^2 \sqrt{2} \int_0^1 \sqrt{\left(v + \frac{9}{16}\right)^2 - \frac{17}{256}} dv = a^2 \sqrt{2} \int_{\frac{9}{16}}^{\frac{25}{16}} \sqrt{t^2 - \frac{17}{256}} dt = \\
&= \frac{a^2}{\sqrt{2}} \left(t \sqrt{t^2 - \frac{17}{256}} - \ln \left(t + \sqrt{t^2 - \frac{17}{256}} \right) \right) \Big|_{\frac{9}{16}}^{\frac{25}{16}} = \\
&= \frac{a^2}{\sqrt{2}} \left(\frac{25}{16} \sqrt{\frac{608}{256}} - \frac{9}{16} \frac{8}{16} - \frac{17}{256} \ln \left(\frac{25}{16} + \sqrt{\frac{608}{256}} \right) + \frac{17}{256} \ln \left(\frac{9}{16} + \frac{8}{16} \right) \right) = \\
&= \frac{a^2}{\sqrt{2}} \left(\frac{25\sqrt{608} - 72}{256} - \frac{17}{256} \ln \left(\frac{25 + \sqrt{608}}{17} \right) \right) = \\
&= \frac{a^2}{256\sqrt{2}} \left(100\sqrt{38} - 72 - 17 \ln \left(\frac{25 + \sqrt{38}}{17} \right) \right)
\end{aligned}$$

4241.2

$$y^2 = 2px, \quad \rho = |y|, \quad 0 \leq x \leq \frac{p}{2}$$

$$m = \int_{\Gamma} |y| dl = 2 \int_0^{\frac{p}{2}} y \sqrt{1 + \left(\frac{y}{p}\right)^2} dy = 2p^2 \int_0^1 u \sqrt{1 + u^2} du = \frac{2}{3} p^2 (1 + u^2)^{3/2} \Big|_0^1 = \frac{2}{3} p^2 (2\sqrt{2} - 1)$$

4243

$$x = a(t - \sin t), y = a(1 - \cos t), 0 \leq t \leq \pi$$

$$dl = a\sqrt{(1 - \cos t)^2 + \sin^2 t} dt = a\sqrt{2(1 - \cos t)} dt = 2a \sin \frac{t}{2} dt$$

$$\int_{\Gamma} dl = 2a \int_0^{\pi} \sin \frac{t}{2} dt = 2a,$$

$$\int_{\Gamma} x dl = 2a^2 \int_0^{\pi} (t - \sin t) \sin \frac{t}{2} dt = 2a^2 \left(-2t \cos \frac{t}{2} \Big|_0^{\pi} + 2 \int_0^{\pi} \cos \frac{t}{2} \left(1 - 2 \sin^2 \frac{t}{2} \right) dt \right) =$$

$$= 2a^2 \left(4 \left(\sin \frac{t}{2} - \frac{2}{3} \sin^3 \frac{t}{2} \right) \Big|_0^{\pi} \right) = 8a^2 \left(1 - \frac{2}{3} \right) = \frac{8}{3} a^2$$

$$\int_{\Gamma} y dl = 2a^2 \int_0^{\pi} (1 - \cos t) \sin \frac{t}{2} dt = 4a^2 \int_0^{\pi} \sin^3 \frac{t}{2} dt = 4a^2 \int_0^{\pi/2} \sin^3 u du = \frac{8}{3} a^2;$$

$$x_0 = y_0 = \frac{4}{3} a$$

4251

$$I = \int_{\Gamma} (x^2 + y^2) dx + (x^2 - y^2) dy, \quad y = 1 - |1 - x|;$$

$$0 \leq x \leq 1 \quad y = x; \quad 1 \leq x \leq 2 \quad y = 2 - x$$

$$I = \int_0^1 2x^2 dx + \int_1^2 2(2 - x)^2 dx = \frac{4}{3}$$

4254

$$I = \int_{\Gamma} \frac{(x + y) dx - (x - y) dy}{x^2 + y^2} = \int_0^{2\pi} (-(\cos t + \sin t) \sin t - (\cos t - \sin t) \cos t) dt = - \int_0^{2\pi} dt = -2\pi$$

$$\Gamma: x^2 + y^2 = a^2$$

4256

$$I = \int_{AB} \sin y dx + \sin x dy \quad A(0, \pi), B(\pi, 0)$$

$$x + y = \pi$$

$$I = \int_0^{\pi} (\sin x - \sin x) dx = 0$$

4264

$$\int_{(1,0)}^{(6,8)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2} \Big|_{(1,0)}^{(6,8)} = 9$$

4273

$$dz = \frac{(x^2 + 2xy + 5y^2)dx + (x^2 - 2xy + y^2)dy}{(x+y)^3}$$

$$dz = \frac{(x^2 + 2xy + 5y^2)dx + (x^2 - 2xy + y^2)dy}{(x+y)^3} =$$

$$= \left(\frac{1}{x+y} + \frac{4y^2}{(x+y)^3} \right) dx + \frac{(x^2 - 2xy + y^2)dy}{(x+y)^3} =$$

$$= d \left(\ln(x+y) - \frac{2y^2}{(x+y)^2} \right) + \left(-\frac{1}{x+y} + \frac{4y}{(x+y)^2} - \frac{4y^2}{(x+y)^3} \right) dy + \frac{(x^2 - 2xy + y^2)dy}{(x+y)^3} =$$

$$= d \left(\ln(x+y) - \frac{2y^2}{(x+y)^2} \right) + \left(-\frac{1}{x+y} + \frac{4xy}{(x+y)^3} \right) dy + \frac{(x^2 - 2xy + y^2)dy}{(x+y)^3} =$$

$$= d \left(\ln(x+y) - \frac{2y^2}{(x+y)^2} \right) + \frac{-x^2 + 2xy - y^2}{(x+y)^3} + \frac{(x^2 - 2xy + y^2)}{(x+y)^3} dy = d \left(\ln(x+y) - \frac{2y^2}{(x+y)^2} \right);$$

$$z = \ln(x+y) - \frac{2y^2}{(x+y)^2}$$

4283

$$\int_{\Gamma} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz,$$

Γ – контур части сферы $x^2 + y^2 + z^2 = 1$, $x, y, z \geq 0$

$$\int_{\Gamma} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz = 3 \int_{\Gamma_1; z=0} (y^2 - z^2)dx + (z^2 - x^2)dy + (x^2 - y^2)dz =$$

$$= 3 \int_{\Gamma_1; z=0} y^2 dx - x^2 dy = \left[\begin{matrix} x = \cos t \\ y = \sin t \end{matrix} \right] = 3 \int_0^{\pi/2} (-\sin^3 t - \cos^3 t) dt = -3 \cdot \frac{4}{3} = -4$$

$$\int_{(0,0,0)}^{(1,2,3)} (2x + y + z^2 + y^2 z)dx + (1 + x + 2xyz + z^3)dy + (3z^2 + 2xz + xy^2 + 3yz^2)dz =$$

$$(x^2 + xy + xz^2 + xy^2 z + y + yz^3 + z^3) \Big|_{(0,0,0)}^{(1,2,3)} = 1 + 2 + 9 + 12 + 2 + 54 + 27 = 107$$

4292

$$du = \frac{(x+y-z)dx + (x+y-z)dy + (x+y+z)dz}{x^2 + y^2 + z^2 + 2xy}$$

$$\begin{aligned} du &= \frac{(x+y-z)dx + (x+y-z)dy + (x+y+z)dz}{(x+y)^2 + z^2} = \\ &= d\left(\frac{1}{2}\ln((x+y)^2 + z^2)\right) + \frac{-z(dx+dy) + (x+y)dz}{(x+y)^2 + z^2} = \\ &= d\left(\frac{1}{2}\ln((x+y)^2 + z^2)\right) + d\left(\operatorname{arctg}\frac{z}{x+y}\right); \\ u &= \frac{1}{2}\ln((x+y)^2 + z^2) + \operatorname{arctg}\frac{z}{x+y} + C \end{aligned}$$