

Занятие 5 продолжение.

Вычислите интегралы с помощью формулы Стокса и непосредственно.

$$1. I = \oint_{\Gamma: x^2+y^2+z^2=3, x+y+z=2} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$$

$$\begin{cases} x^2 + y^2 + z^2 = 3 \\ x + y + z = 2 \end{cases} \begin{cases} z = 2 - x - y \\ x^2 + y^2 + (2 - x - y)^2 = 3 \end{cases}$$

$$2x^2 + 2y^2 + 4 - 4x - 4y + 2xy =$$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(u - v) \\ y = \frac{1}{\sqrt{2}}(u + v) \end{cases} \begin{cases} 2u^2 + 2v^2 + u^2 - v^2 - 4\sqrt{2}u + 1 = 0 \\ 3u^2 + v^2 - 4\sqrt{2}u + 1 = 0 \\ 3\left(u - \frac{2\sqrt{2}}{3}\right)^2 + v^2 = \frac{5}{3} \end{cases} \begin{cases} u = \frac{2\sqrt{2}}{3} + \frac{\sqrt{5}}{3} \cos t \\ v = \frac{\sqrt{5}}{\sqrt{3}} \sin t \end{cases}$$

Решение_1.

$$\begin{aligned} I &= \oint_{\Gamma: x^2+y^2+z^2=3, x+y+z=2} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz = [dz = -dx - dy] = \\ &= \oint_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy - (x^2 - y^2)(dx + dy) = \\ &= \oint_{\Gamma} (-x^2 + 2y^2 - z^2) dx + (z^2 - 2x^2 + y^2) dy = \\ &= \oint_{\Gamma} (-x^2 + 2y^2 - 3 + x^2 + y^2) dx + (3 - x^2 - y^2 - 2x^2 + y^2) dy = \\ &= \oint_{\Gamma} (3y^2 - 3) dx + (3 - 3x^2) dy = 3 \oint_{\Gamma} (y^2 dx - x^2 dy) \\ &= 3 \oint_{\Gamma_1} \left(\frac{1}{2\sqrt{2}}(u+v)^2 (du - dv) - \frac{1}{2\sqrt{2}}(u-v)^2 (du + dv) \right) = \\ &= \frac{3}{2\sqrt{2}} \oint_{\Gamma_1} \left((u^2 + v^2)(-2dv) + 2uv2du \right) = \frac{3}{2\sqrt{2}} \oint_{\Gamma_1} (-2u^2 dv + 4uv du) = \\ &= \frac{3}{2\sqrt{2}} \int_0^{2\pi} \left(-2 \left(\frac{2\sqrt{2}}{3} + \frac{\sqrt{5}}{3} \cos t \right)^2 \frac{\sqrt{5}}{\sqrt{3}} \cos t + 4 \left(\frac{2\sqrt{2}}{3} + \frac{\sqrt{5}}{3} \cos t \right) \frac{\sqrt{5}}{\sqrt{3}} \sin t \frac{\sqrt{5}}{\sqrt{3}} \cos t \right) dt = \\ &= -4 \frac{3}{2\sqrt{2}} \frac{2\sqrt{2}}{3} \frac{\sqrt{5}}{3} \frac{\sqrt{5}}{\sqrt{3}} \pi = -\frac{40}{3\sqrt{3}} \pi. \end{aligned}$$

Решение_2.

$$I = \frac{1}{\sqrt{3}} \iint_{\Sigma} \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} d\sigma = -\frac{4}{\sqrt{3}} \iint_{\Sigma} (x + y + z) d\sigma =$$

$$= -\frac{8}{\sqrt{3}} \iint_{\Sigma} d\sigma = -8 \iint_E dx dy = -8 \frac{\sqrt{5}}{3} \frac{\sqrt{5}}{\sqrt{3}} \pi = -\frac{40}{3\sqrt{3}} \pi$$

2.

$$I = \oint_{\Gamma} (x + z) dx + (x - y) dy + x dz, \Gamma - \text{эллипс } \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, z = c,$$

ориентированный положительно относительно вектора $(0, 0, 1)$.

Решение_1.

$$\oint_{\Gamma} (x + z) dx + (x - y) dy + x dz = \begin{bmatrix} x = a \cos t \\ y = b \sin t \\ z = c \end{bmatrix} =$$

$$= \int_0^{2\pi} ((a \cos t + c)(-a \sin t) + (a \cos t - b \sin t)b \cos t) dt = \pi ab$$

Решение_2.

$$I = \oint_{\Gamma} (x + z) dx + (x - y) dy + x dz = \iint_{\Sigma: z=c, \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1} \begin{vmatrix} 0 & 0 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x + z & x - y & x \end{vmatrix} d\sigma = \iint_{\Sigma} d\sigma = \pi ab$$

3.

$$I = \int_{\Gamma} x dx + (x + y) dy + (x + y + z) dz =$$

$$x = a \sin t, y = a \cos t, z = a(\sin t + \cos t), t \in [0, 2\pi]$$

Решение_1.

$$\begin{aligned}
 I &= \int_{\Gamma} xdx + (x+y)dy + (x+y+z)dz = \\
 &= \int_{\Gamma} xdy + (x+y)dz = \int_{\Gamma} xdy + zdz = \int_{\Gamma} xdy = \int_0^{2\pi} a \sin t (-a \cos t) dt = -\pi a^2
 \end{aligned}$$

Решение_2.

$$z = x + y,$$

$$I = \frac{1}{\sqrt{3}} \iint_{\Sigma: z=x+y, x^2+y^2 \leq a^2} \begin{vmatrix} 1 & 1 & -1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & x+y & x+y+z \end{vmatrix} d\sigma = \frac{1}{\sqrt{3}} \iint_{\Sigma} (1-1-1) d\sigma = -\frac{1}{\sqrt{3}} \iint_{\Sigma} d\sigma = -\iint_E dx dy = -\pi a^2.$$

4.

$$I = \int_{\Gamma} ydx + zdy + xdz = \pi\sqrt{3}a^2,$$

Γ – окружность $x^2 + y^2 + z^2 = a^2$, $x + y + z = 0$

Решение_1.

$$\begin{cases} x^2 + y^2 + z^2 = a^2, \\ x + y + z = 0, \end{cases} \begin{cases} z = -x - y, \\ x^2 + y^2 + z^2 = a^2, \end{cases}$$

$$x^2 + y^2 + (x+y)^2 = a^2, 2x^2 + 2y^2 + 2xy = a^2,$$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(u-v), \\ y = \frac{1}{\sqrt{2}}(u+v), \end{cases}$$

$$2u^2 + 2v^2 + u^2 - v^2 = a^2, 3u^2 + v^2 = a^2.$$

$$\begin{cases} u = \frac{a}{\sqrt{3}} \cos t, \\ v = a \sin t, \end{cases} \begin{cases} x = \frac{a}{\sqrt{2}} \left(\frac{1}{\sqrt{3}} \cos t - \sin t \right), \\ y = \frac{a}{\sqrt{2}} \left(\frac{1}{\sqrt{3}} \cos t + \sin t \right), \\ z = -\frac{\sqrt{2}a}{\sqrt{3}} \cos t \end{cases}, \begin{cases} x = \frac{\sqrt{2}a}{\sqrt{3}} \cos \left(t + \frac{\pi}{3} \right), \\ y = \frac{\sqrt{2}a}{\sqrt{3}} \cos \left(t - \frac{\pi}{3} \right), \\ z = -\frac{\sqrt{2}a}{\sqrt{3}} \cos t \end{cases}$$

$$I = \frac{a^2}{2} \int_0^{2\pi} \left(\left(\frac{1}{\sqrt{3}} \cos t + \sin t \right) \left(-\frac{1}{\sqrt{3}} \sin t - \cos t \right) - \frac{2}{\sqrt{3}} \cos t \left(-\frac{1}{\sqrt{3}} \sin t + \cos t \right) + \left(\frac{1}{\sqrt{3}} \cos t - \sin t \right) \frac{2}{\sqrt{3}} \sin t \right) dt =$$

$$= \frac{a^2}{2} \pi \left(-\frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} - \frac{2}{\sqrt{3}} \right) = -\sqrt{3} a^2 \pi$$

Решение_2.

$$I = \int_{\Gamma} y dx + z dy + x dz = \frac{1}{\sqrt{3}} \iint_{\Sigma} \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & z & x \end{vmatrix} d\sigma = \frac{1}{\sqrt{3}} \iint_{\Sigma} (-1-1-1) d\sigma = -\sqrt{3} \iint_{\Sigma} d\sigma = -\sqrt{3} \pi a^2$$

5.

$\int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz$, Γ — кривая пересечения поверхности куба

$|x|, |y|, |z| \leq a$ плоскостью $x + y + z = \frac{3a}{2}$

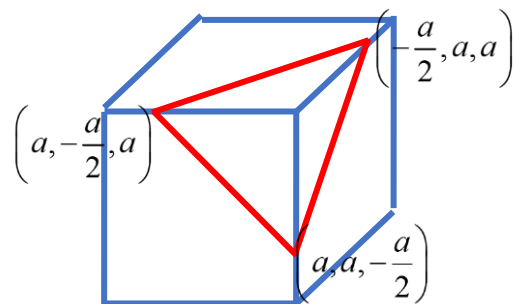
Решение_1.

$$x + y = \frac{a}{2}, z = a;$$

$$I = 3 \int_{\left(-\frac{a}{2}, a, a\right)}^{\left(a, -\frac{a}{2}, a\right)} (y^2 - a^2) dx + (a^2 - x^2) dy = 3 \int_{\left(-\frac{a}{2}, a, a\right)}^{\left(a, -\frac{a}{2}, a\right)} (y^2 - a^2)(-dy) + (a^2 - x^2)(-dx)$$

$$= 3 \left(-\frac{y^3}{3} + a^2 y - a^2 x + \frac{x^3}{3} \right) \Big|_{\left(-\frac{a}{2}, a, a\right)}^{\left(a, -\frac{a}{2}, a\right)} = 3 \left(\frac{9a^3}{3 \cdot 8} - \frac{3}{2} a^3 - \frac{3}{2} a^3 + \frac{9a^3}{3 \cdot 8} \right) =$$

$$= 3a^3 \left(\frac{3}{4} - 3 \right) = -\frac{27}{4} a^3$$



Решение_2

$$I = \int_{\Gamma} (y^2 - z^2) dx + (z^2 - x^2) dy + (x^2 - y^2) dz =$$

$$= \frac{1}{\sqrt{3}} \iint_{\Sigma} \begin{vmatrix} 1 & 1 & 1 \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 - z^2 & z^2 - x^2 & x^2 - y^2 \end{vmatrix} d\sigma = -\frac{2}{\sqrt{3}} \iint_{\Sigma} (y + z + z + x + x + y) d\sigma =$$

$$= -\frac{2}{\sqrt{3}} \iint_{\Sigma} (2x + 2y + 2z) d\sigma = -\frac{4}{\sqrt{3}} \iint_{\Sigma} (x + y + z) d\sigma = -\frac{4 \cdot 3a}{2\sqrt{3}} \iint_{\Sigma} d\sigma = -\frac{4\sqrt{3}a}{2} \left(\frac{3}{\sqrt{2}} a \right)^2 \frac{\sqrt{3}}{4} = -\frac{27a^3}{4},$$