ЗАНЯТИЕ 1 КРИВОЛИНЕЙНЫЕ ИНТЕГРАЛЫ

4221

$$\int_{\Gamma}(x+y)\,dl$$

$$O(0,0),\ A(1,0),\ B(0,1)$$

$$\int_{OA}(x+y)\,dl=\int^{1}xdx=\frac{1}{2},\ \int_{AB}(x+y)\,dl=\int_{AB}dl=\sqrt{2},$$

Ответ: $\sqrt{2}+1.$

4222

$$I=\int_{\Gamma}y^2dl;\ x=a(t-\sin t),\ y=a(1-\cos t);$$

$$dl=a\sqrt{(1-\cos t)^2+\sin^2 t}\,dt=a\sqrt{2-2\cos t}\,dt=2a\sin\frac{t}{2}dt$$

$$I=\int_0^{2\pi}a^2(1-\cos t)^2\,2a\sin\frac{t}{2}dt=8a^3\int_0^{2\pi}\sin^5\frac{t}{2}dt=16a^3\int_0^{\pi}\sin^5 udu=32a^3\frac{4!!}{5!!}=\frac{256}{15}a^3$$

4225

$$I=\int_{\Gamma}\left(x^{4/3}+y^{4/3}\right)dl,\ \Gamma:\ x^{2/3}+y^{2/3}=a^{2/3};$$

$$x=a\cos^3 t,\ \ y=a\sin^3 t;$$

$$dl=a\sqrt{\left(-3\cos^2 t\sin t\right)^2+\left(-3^2\sin t\cos t\right)^2}\,dt=3a\cos t\sin tdt,\ \ 0\le t\le\pi/2;$$

$$I=12a^{7/3}\int_0^{\pi/2}\left(\cos^4 t+\sin^4 t\right)\cos t\sin tdt=24a^{7/3}\int_0^{\pi/2}\cos t\sin^5 tdt=24a^{7/3}\cdot\frac{1}{6}=4a^{7/3}$$

4229

$$I=\int_{\Gamma}\sqrt{x^2+y^2}\,dl,\ \Gamma:\ \ x^2+y^2=ax$$

$$x=\frac{a}{2}(1+\cos t),\ y=\frac{a}{2}\sin t;\ dl=\frac{a}{2}dt;$$

$$I=\frac{a^2}{2\sqrt{2}}\int_0^{2\pi}\sqrt{1+\cos t}\,dt=\frac{a^2}{\sqrt{2}}\int_0^{\pi}\sqrt{1+\cos t}\,dt=a^2\int_0^{\pi}\cos\frac{t}{2}dt=2a^2$$

4232

$$x=e^{-t}\cos t,\ y=e^{-t}\sin t,\ z=e^{-t};$$

$$dl=\sqrt{\left(e^{-t}\left(-\cos t-\sin t\right)\right)^2+\left(e^{-t}\left(-\sin t+\cos t\right)\right)^2+e^{-2t}}dt=e^{-t}\sqrt{3}$$

$$L=\sqrt{3}\int_0^{+\infty}e^{-t}dt=\sqrt{3}$$

4234

$$\left(x-y\right)^2=a\left(x+y\right),\ x^2-y^2=\frac{9}{8}z^2;$$

$$O\left(0,0,0\right),\ A\left(x_0,y_0,z_0\right)$$

$$x=\frac{1}{\sqrt{2}}\left(u-v\right),y=\frac{1}{\sqrt{2}}\left(u+v\right),$$

$$2v^2=\sqrt{2}au,\ -2uv=\frac{9}{8}z^2;$$

$$u=\frac{\sqrt{2}v^2}{a},\ uv=-\frac{9}{16}z^2,\ \frac{\sqrt{2}v^3}{a}=-\frac{9}{16}z^2,$$

$$v^3=-\frac{9a}{16\sqrt{2}}z^2;$$

$$v=-\left(\frac{9a}{16\sqrt{2}}\right)^{1/3}z^{2/3}=-\frac{3^{2/3}a^{1/3}}{2^{3/2}}z^{2/3},u=\frac{\sqrt{2}}{a^{1/3}}\left(\frac{9}{16\sqrt{2}}\right)^{2/3}z^{4/3}=\frac{3^{4/3}}{a^{1/3}2^{5/2}}z^{4/3};$$

$$dl=\sqrt{1+\left(-\frac{3^{2/3}a^{1/3}}{2^{3/2}}\frac{2}{3}z^{-1/3}\right)^2+\left(\frac{3^{4/3}}{a^{1/3}2^{5/2}}\frac{4}{3}z^{1/3}\right)}dz=\sqrt{1+\left(-\frac{a^{1/3}}{2^{1/2}3^{1/3}}z^{-1/3}\right)^2+\left(\frac{3^{1/3}}{a^{1/3}2^{1/2}}z^{1/3}\right)^2}dz=$$

$$=\left(\frac{a^{1/3}}{2^{1/2}3^{1/3}}z^{-1/3}+\frac{3^{1/3}}{a^{1/3}2^{1/2}}z^{1/3}\right)dz;$$

$$L=\int_0^{z_0}\left(\frac{a^{1/3}}{2^{1/2}3^{1/3}}z^{-1/3}+\frac{3^{1/3}}{a^{1/3}2^{1/2}}z^{1/3}\right)dz=\frac{3^{2/3}a^{1/3}}{2^{3/2}3^{1/3}}z_0^{2/3}+\frac{3^{4/3}}{a^{1/3}2^{5/2}}z_0^{4/3}$$

4238

$$\int_\Gamma x^2dl,\ \Gamma:x^2+y^2+z^2=a^2,x+y+z=0$$

$$\int_\Gamma x^2dl=\frac{1}{3}\int_\Gamma\left(x^2+y^2+z^2\right)dl=\frac{a^2}{3}\int_\Gamma dl=\frac{2\pi a^2}{3}$$

$$z = -(x+y)$$
$$x^2 + y^2 + (x+y)^2 = a^2$$
$$x^2 + xy + y^2 = \frac{a^2}{2}$$
$$\begin{cases} x = \frac{1}{\sqrt{2}}(u-v) \\ y = \frac{1}{\sqrt{2}}(u+v) \end{cases}$$
$$u^2 + v^2 + \frac{1}{2}\left(u^2 - v^2\right) = \frac{a^2}{2}$$
$$3u^2 + v^2 = a^2$$
$$\begin{cases} x^2 + y^2 + z^2 = a^2 \\ x + y + z = 0 \end{cases} \quad \begin{cases} u = \frac{a}{\sqrt{3}}\cos t \\ v = a\sin t \end{cases} \quad \begin{cases} x = \frac{a}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\cos t - \sin t\right) = \frac{\sqrt{2}a}{\sqrt{3}}\cos\left(t + \frac{\pi}{3}\right) \\ y = \frac{a}{\sqrt{2}}\left(\frac{1}{\sqrt{3}}\cos t + \sin t\right) = \frac{\sqrt{2}a}{\sqrt{3}}\cos\left(t - \frac{\pi}{3}\right) \\ z = -\frac{\sqrt{2}a}{\sqrt{3}}\cos t \end{cases}$$
$$dl = a\sqrt{\frac{1}{2}\left(-\frac{1}{\sqrt{3}}\sin t - \cos t\right)^2 + \frac{1}{2}\left(-\frac{1}{\sqrt{3}}\sin t + \cos t\right)^2 + \frac{2}{3}\sin^2 t}\,dt = adl$$
$$I = \int_0^{2\pi} \frac{2a^3}{3}\cos^2\left(t + \frac{\pi}{3}\right)dt = \frac{2\pi a^3}{3}$$

4240

$$\int_\Gamma zdl,\ \Gamma : x^2 + y^2 = z^2, y^2 = ax \quad a, x, y, z > 0$$
$$x^2 + y^2 = z^2, y^2 = ax$$
$$x = \frac{y^2}{a},$$
$$\frac{y^4}{a^2} + y^2 = z^2,\ z = \frac{\sqrt{y^4 + a^2y^2}}{a}$$

$$dl=\sqrt{\left(\frac{2y}{a}\right)^2+1+\left(\frac{2y^3+a^2y}{a\sqrt{y^4+a^2y^2}}\right)^2}dy=\sqrt{\left(\frac{2y}{a}\right)^2+1+\left(\frac{2y^2+a^2}{a\sqrt{y^2+a^2}}\right)^2}dy=$$

$$=\sqrt{\frac{4y^2}{a^2}+1+\frac{4y^4+4a^2y^2+a^4}{a^2\left(y^2+a^2\right)}}dy=\sqrt{\frac{8y^4+9a^2y^2+2a^4}{a^2\left(y^2+a^2\right)}}dy$$

$$\int_\Gamma zdl=\int_0^a\frac{\sqrt{y^4+a^2y^2}}{a}\sqrt{\frac{8y^4+9a^2y^2+2a^4}{a^2\left(y^2+a^2\right)}}dy=\frac{1}{a^2}\int_0^a y\sqrt{8y^4+9a^2y^2+2a^4}dy=$$

$$=\frac{1}{2a^2}\int_0^{a^2}\sqrt{8u^2+9a^2u+2a^4}du=\frac{a^2}{2}\int_0^1\sqrt{8v^2+9v+2}dv=a^2\sqrt{2}\int_0^1\sqrt{v^2+\frac{9}{8}v+\frac{1}{4}}dv=$$

$$=a^2\sqrt{2}\int_0^1\sqrt{\left(v+\frac{9}{16}\right)^2-\frac{17}{256}}dv=a^2\sqrt{2}\int_{\frac{9}{16}}^{\frac{25}{16}}\sqrt{t^2-\frac{17}{256}}dt=$$

$$=\frac{a^2}{\sqrt{2}}\left(t\sqrt{t^2-\frac{17}{256}}-\ln\left(t+\sqrt{t^2-\frac{17}{256}}\right)\right)\Bigg|_{9/16}^{25/16}=$$

$$=\frac{a^2}{\sqrt{2}}\left(\frac{25}{16}\sqrt{\frac{608}{256}}-\frac{9}{16}\frac{8}{16}-\frac{17}{256}\ln\left(\frac{25}{16}+\sqrt{\frac{608}{256}}\right)+\frac{17}{256}\ln\left(\frac{9}{16}+\frac{8}{16}\right)\right)=$$

$$=\frac{a^2}{\sqrt{2}}\left(\frac{25\sqrt{608}-72}{256}-\frac{17}{256}\ln\left(\frac{25+\sqrt{608}}{17}\right)\right)=$$

$$=\frac{a^2}{256\sqrt{2}}\left(100\sqrt{38}-72-17\ln\left(\frac{25+\sqrt{38}}{17}\right)\right)$$

4241.2

$$y^2=2px,\ \ \rho=|y|,\ 0\le x\le\frac{p}{2}$$

$$m=\int_\Gamma|y|dl=2\int_0^p y\sqrt{1+\left(\frac{y}{p}\right)^2}dy=2p^2\int_0^1 u\sqrt{1+u^2}du=\frac{2}{3}p^2\left(1+u^2\right)^{3/2}\Big|_0^1=\frac{2}{3}p^2\left(2\sqrt{2}-1\right)$$

4243

$$x = a(t - \sin t), y = a(1 - \cos t),\ 0 \le t \le \pi$$

$$dl = a\sqrt{(1-\cos t)^2 + \sin^2 t}\,dt = a\sqrt{2(1-\cos t)}\,dt = 2a\sin\frac{t}{2}dt$$

$$\int_\Gamma dl = 2a\int_0^\pi \sin\frac{t}{2}dt = 2a,$$

$$\int_\Gamma xdl = 2a^2\int_0^\pi (t - \sin t)\sin\frac{t}{2}dt = 2a^2\left(-2t\cos\frac{t}{2}\Big|_0^\pi + 2\int_0^\pi \cos\frac{t}{2}\left(1 - 2\sin^2\frac{t}{2}\right)dt\right) =$$

$$= 2a^2\, 4\left(\sin\frac{t}{2} - \frac{2}{3}\sin^3\frac{t}{2}\right)\Big|_0^\pi = 8a^2\left(1 - \frac{2}{3}\right) = \frac{8}{3}a^2$$

$$\int_\Gamma ydl = 2a^2\int_0^\pi (1 - \cos t)\sin\frac{t}{2}dt = 4a^2\int_0^\pi \sin^3\frac{t}{2}dt = 4a^2\int_0^{\pi/2}\sin^3 u\,du = \frac{8}{3}a^2;$$

$$x_0 = y_0 = \frac{4}{3}a$$

4251

$$I = \int_\Gamma (x^2 + y^2)dx + (x^2 - y^2)dy,\ \ y = 1 - |1 - x|;$$

$$0 \le x \le 1\ \ \ y = x;\ \ \ 1 \le x \le 2\ \ \ y = 2 - x$$

$$I = \int_0^1 2x^2 dx + \int_1^2 2(2 - x)^2\,dx = \frac{4}{3}$$

4254

$$I = \int_\Gamma \frac{(x+y)dx - (x-y)dy}{x^2 + y^2} = \int_0^{2\pi}\left(-(\cos t + \sin t)\sin t - (\cos t - \sin t)\cos t\right)dt = -\int_0^{2\pi} dt = -2\pi$$

$$\Gamma : x^2 + y^2 = a^2$$

4256

$$I = \int_{AB} \sin y dx + \sin x dy \qquad A(0,\pi), B(\pi,0)$$

$$x + y = \pi$$

$$I = \int_0^\pi (\sin x - \sin x)dx = 0$$

4264

$$\int_{(1,0)}^{(6,8)} \frac{xdx + ydy}{\sqrt{x^2 + y^2}} = \sqrt{x^2 + y^2}\Big|_{(1,0)}^{(6,8)} = 9$$

4273

$$dz=\frac{\left(x^2+2xy+5y^2\right)dx+\left(x^2-2xy+y^2\right)dy}{\left(x+y\right)^3}$$

$$dz=\frac{\left(x^2+2xy+5y^2\right)dx+\left(x^2-2xy+y^2\right)dy}{\left(x+y\right)^3}=$$

$$=\left(\frac{1}{x+y}+\frac{4y^2}{\left(x+y\right)^3}\right)dx+\frac{\left(x^2-2xy+y^2\right)dy}{\left(x+y\right)^3}=$$

$$=d\left(\ln\left(x+y\right)-\frac{2y^2}{\left(x+y\right)^2}\right)+\left(-\frac{1}{x+y}+\frac{4y}{\left(x+y\right)^2}-\frac{4y^2}{\left(x+y\right)^3}\right)dy+\frac{\left(x^2-2xy+y^2\right)dy}{\left(x+y\right)^3}=$$

$$=d\left(\ln\left(x+y\right)-\frac{2y^2}{\left(x+y\right)^2}\right)+\left(-\frac{1}{x+y}+\frac{4xy}{\left(x+y\right)^3}\right)dy+\frac{\left(x^2-2xy+y^2\right)}{\left(x+y\right)^3}dy=$$

$$=d\left(\ln\left(x+y\right)-\frac{2y^2}{\left(x+y\right)^2}\right)+\frac{-x^2+2xy-y^2}{\left(x+y\right)^3}+\frac{\left(x^2-2xy+y^2\right)}{\left(x+y\right)^3}dy=d\left(\ln\left(x+y\right)-\frac{2y^2}{\left(x+y\right)^2}\right);$$

$$z=\ln\left(x+y\right)-\frac{2y^2}{\left(x+y\right)^2}$$

4283

$$\int\limits_{\Gamma}\left(y^2-z^2\right)dx+\left(z^2-x^2\right)dy+\left(x^2-y^2\right)dz,$$

Γ − контур части сферы $x^2+y^2+z^2=1,\ x,y,z\geq 0$

$$\int\limits_{\Gamma}\left(y^2-z^2\right)dx+\left(z^2-x^2\right)dy+\left(x^2-y^2\right)dz=3\int\limits_{\Gamma_1:z=0}\left(y^2-z^2\right)dx+\left(z^2-x^2\right)dy+\left(x^2-y^2\right)dz=$$

$$=3\int\limits_{\Gamma_1:z=0}y^2dx-x^2dy=\begin{bmatrix}x=\cos t\\ y=\sin t\end{bmatrix}=3\int\limits_0^{\pi/2}\left(-\sin^3 t-\cos^3 t\right)dt=-3\cdot\frac{4}{3}=-4$$

$$\int\limits_{(0,0,0)}^{(1,2,3)}\left(2x+y+z^2+y^2z\right)dx+\left(1+x+2xyz+z^3\right)dy+\left(3z^2+2xz+xy^2+3yz^2\right)dz=$$

$$\left(x^2+xy+xz^2+xy^2z+y+yz^3+z^3\right)\Big|_{(0,0,0)}^{(1,2,3)}=1+2+9+12+2+54+27=107$$

4292

$$du=\frac{\left(x+y-z\right)dx+\left(x+y-z\right)dy+\left(x+y+z\right)dz}{x^2+y^2+z^2+2xy}$$

$$du = \frac{(x+y-z)\,dx + (x+y-z)\,dy + (x+y+z)\,dz}{(x+y)^2 + z^2} =$$

$$= d\left(\frac{1}{2}\ln\left((x+y)^2 + z^2\right)\right) + \frac{-z\left(dx+dy\right) + (x+y)\,dz}{(x+y)^2 + z^2} =$$

$$= d\left(\frac{1}{2}\ln\left((x+y)^2 + z^2\right)\right) + d\left(\operatorname{arctg}\frac{z}{x+y}\right);$$

$$u = \frac{1}{2}\ln\left((x+y)^2 + z^2\right) + \operatorname{arctg}\frac{z}{x+y} + C$$