Интегралы Эйлера

1⁰. B -функция Эйлера

Определение

$$\mathrm{B}(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\,dx,\ p,q>0$$

Симметрия

$$\mathrm{B}(p,q)=\mathrm{B}(q,p)$$

Другое аналитическое выражение для B -функции

$$\mathrm{B}(p,q)=\int_0^{+\infty}\frac{x^{p-1}}{(1+x)^{p+q}}\,dx$$

2⁰. Γ -функция Эйлера

$$\Gamma(s)=\int\limits_0^{+\infty} x^{s-1}e^{-x}dx,\ s>0.$$

Рекуррентная формула

$$\Gamma(s+1)=s\Gamma(s),$$

$$\Gamma(n+1)=n!.$$

$$\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$$

$$\Gamma\left(\frac{2n+1}{2}\right)=\frac{1}{2}\cdot\frac{3}{2}\cdots\frac{2n-1}{2}\sqrt{\pi}$$

Формула дополнения для Γ -функции

$$\Gamma(s)\Gamma(1-s)=\frac{\pi}{\sin\pi s},\ s\in(0,1).$$

3⁰. Связь B и Γ функций

$$\mathrm{B}(p,q)=\frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

3846 $\displaystyle\int\limits_0^{+\infty}\frac{dx}{1+x^3}=\frac{2\pi}{3\sqrt{3}}$

$$\int_0^{+\infty}\frac{dx}{1+x^3}=\left[y=x^3,\ x=y^{1/3}\right]=\frac{1}{3}\int_0^{+\infty}\frac{y^{1/3-1}dy}{1+y}=\frac{1}{3}\mathrm{B}\left(\frac{1}{3},\ \frac{2}{3}\right)=$$

$$=\frac{1}{3}\cdot\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right)}{\Gamma(1)}=\frac{1}{3}\frac{\pi}{\sin\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}$$

3856 $\displaystyle\int_0^{\pi/2}\sin^{\alpha-1}x\cos^{\beta-1}x dx\ =\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)}$

Выполним замену переменной интегрирования по формуле $u=\sin^2 x,\ du=2\sin x\cos x$

$$\int_0^{\pi/2}\sin^{\alpha-1}x\cos^{\beta-1}x dx=\frac{1}{2}\int_0^{\pi/2}\sin^{\alpha-2}x\cos^{\beta-2}x\ 2\sin x\cos x dx=$$

$$=\frac{1}{2}\int_0^1 u^{\frac{\alpha}{2}-1}(1-u)^{\frac{\beta}{2}-1}du=\frac{1}{2}\mathrm{B}\left(\frac{\alpha}{2},\frac{\beta}{2}\right)=\frac{\Gamma\left(\frac{\alpha}{2}\right)\Gamma\left(\frac{\beta}{2}\right)}{2\Gamma\left(\frac{\alpha+\beta}{2}\right)}$$

$$\int_0^{\pi/2}\sin^{5/6}x\cos^{7/6}x dx=\frac{\pi\left(\sqrt{3}+1\right)}{12\sqrt{2}}$$

$$\int_0^{\pi/2}\sin^{5/6}x\cos^{7/6}x dx=\frac{1}{2}\int_0^{\pi/2}\sin^{-1/6}x\cos^{1/6}x\ 2\sin x\cos x dx=$$

$$=\frac{1}{2}\int_0^1 u^{-1/12}(1-u)^{1/12}du=\frac{1}{2}\mathrm{B}\left(\frac{11}{12},\frac{13}{12}\right)=\frac{\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{13}{12}\right)}{2\Gamma(2)}=$$

$$=\frac{\Gamma\left(\frac{11}{12}\right)\Gamma\left(\frac{1}{12}\right)\frac{1}{12}}{2}=\frac{\pi}{24\sin\frac{\pi}{12}}=\frac{\pi}{24\,\frac{\sqrt{3}-1}{2\sqrt{2}}}=\frac{\pi\left(\sqrt{3}+1\right)}{12\sqrt{2}}$$

$$\int_3^5\frac{(x-3)^{7/4}(5-x)^{1/4}}{(x+1)^4}dx\ =\frac{7\pi}{2304\ 6^{3/4}}$$

$$I=\int_3^5 \frac{(x-3)^{7/4}(5-x)^{1/4}}{(x+1)^{9/2}}dx=\int_3^5\left(\frac{x-3}{x+1}\right)^{7/4}\left(\frac{5-x}{x+1}\right)^{1/4}\frac{dx}{(x+1)^2},$$

$$u=\frac{x-3}{x+1},\ v=\frac{5-x}{x+1},\ du=\frac{4}{(x+1)^2}dx,\ dv=-\frac{6}{(x+1)^2}dx,$$

$$3u+2v=1;$$

$$I=\frac{1}{4}\int_0^{1/3}u^{7/4}\left(\frac{1-3u}{2}\right)^{1/4}du=\frac{1}{3}\cdot\frac{1}{3^{7/4}}\cdot\frac{1}{2^{9/4}}\int_0^1 t^{7/4}(1-t)^{1/4}dt=$$

$$=\frac{1}{3^{11/4}}\cdot\frac{1}{2^{9/4}}\mathrm{B}\left(\frac{11}{4},\frac{5}{4}\right)=\frac{1}{3^{11/4}}\cdot\frac{1}{2^{9/4}}\cdot\frac{\Gamma\left(\frac{11}{4}\right)\Gamma\left(\frac{5}{4}\right)}{\Gamma(4)}=\frac{1}{3^{11/4}}\cdot\frac{1}{2^{9/4}}\cdot\frac{\Gamma\left(\frac{3}{4}\right)\frac{3}{4}\cdot\frac{7}{4}\Gamma\left(\frac{1}{4}\right)\frac{1}{4}}{\Gamma(4)}=$$

$$\frac{1}{3^{11/4}}\cdot\frac{1}{2^{9/4}}\cdot\frac{21\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right)}{2^6\cdot 3!}=\frac{1}{3^{11/4}}\cdot\frac{1}{2^{33/4}}\cdot\frac{21\pi}{6\sin\frac{\pi}{4}}=\frac{1}{3^{11/4}}\cdot\frac{1}{2^{33/4}}\cdot\frac{7\pi}{\sqrt{2}}=\frac{1}{3^{11/4}}\cdot\frac{7\pi}{2^{35/4}}=\frac{7\pi}{9\cdot 256\cdot 6^{3/4}}=\frac{7\pi}{2304\cdot 6^{3/4}}$$

3864 в) $\int_0^{+\infty}\frac{\ln^2 x}{1+x^4}dx=\frac{3\pi^3}{32\sqrt{2}}$

Рассмотрим интеграл, зависящий от параметра

$$I(\alpha)=\int_0^{+\infty}\frac{x^{\alpha-1}}{1+x^4}dx=\left[y=x^4\right]=\frac{1}{4}\int_0^{+\infty}\frac{y^{\frac{\alpha-1}{4}-\frac{3}{4}}}{1+y}dy=\frac{1}{4}\int_0^{+\infty}\frac{y^{\frac{\alpha}{4}-1}}{1+y}dy=\frac{1}{4}\mathrm{B}\left(\frac{\alpha}{4},1-\frac{\alpha}{4}\right)=\frac{\pi}{4\sin\frac{\pi\alpha}{4}}.$$

Проведем двукратное дифференцирование:

$$I'(\alpha)=\int_0^{+\infty}\frac{x^{\alpha-1}\ln x}{1+x^4}dx=-\frac{\pi^2\cos\frac{\pi\alpha}{4}}{16\sin^2\frac{\pi\alpha}{4}},$$

$$I''(\alpha)=\int_0^{+\infty}\frac{x^{\alpha-1}\ln^2 x}{1+x^4}dx=\frac{\pi^3}{64\sin\frac{\pi\alpha}{4}}+\frac{2\pi^3\cos^2\frac{\pi\alpha}{4}}{64\sin^3\frac{\pi\alpha}{4}}$$

Положим $\alpha=0$: $\int_0^{+\infty}\frac{\ln^2 x}{1+x^4}dx=\frac{\pi^3\sqrt{2}}{64}+\frac{2\pi^3\sqrt{2}}{64}=\frac{3\pi^3}{32\sqrt{2}}$

3868 $\int\limits_0^1 \ln\Gamma(x)\,dx = \ln\sqrt{2\pi}$

$$I = \int\limits_0^1 \ln\Gamma(x)\,dx = \int\limits_0^1 \ln\Gamma(1-x)\,dx,$$

$$2I = \int\limits_0^1 \ln\left(\Gamma(x)\Gamma(1-x)\right)dx = \int\limits_0^1 \ln\frac{\pi}{\sin\pi x}\,dx,$$

$$2I = \int\limits_0^1 \ln\frac{\pi}{\sin\pi x}\,dx = 2\int\limits_0^{1/2} \ln\frac{\pi}{\sin\pi x}\,dx = 2\int\limits_0^{1/2} \ln\frac{\pi}{\cos\pi x}\,dx,$$

$$2I = \int\limits_0^{1/2} \ln\frac{\pi^2}{\sin\pi x\cos\pi x}\,dx = \int\limits_0^{1/2} \ln\frac{2\pi^2}{\sin 2\pi x}\,dx = \frac{1}{2}\ln 2\pi + \int\limits_0^{1/2} \ln\frac{\pi}{\sin 2\pi x}\,dx = \frac{1}{2}\ln 2\pi + I,$$

$$I = \ln\sqrt{2\pi}$$

Домашнее задание. 3847, 3848, 3851, 3863, 3864 б), 3872

Интегралы Эйлера (продолжение)

1. $A = \int\limits_{-1}^{1} \ln\frac{1+x}{1-x}\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}} = \frac{2\pi^2}{3}$

Рассмотрим интеграл, зависящий от параметра

$$I(\alpha) = \int\limits_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\alpha}\frac{dx}{1+x} = \int\limits_{-1}^{1}(1+x)^{\alpha-1}(1-x)^{-\alpha}\,dx = \left[y = \frac{1+x}{2}, x = 2y-1\right] =$$

$$\int\limits_0^1 y^{\alpha-1}(1-y)^{-\alpha}\,dx = \mathrm{B}(\alpha, 1-\alpha) = \frac{\pi}{\sin\pi\alpha};$$

Продифференцируем полученное равенство:

$$I'(\alpha) = \int\limits_{-1}^{1}\left(\frac{1+x}{1-x}\right)^{\alpha}\ln\frac{1+x}{1-x}\frac{dx}{1+x} = -\frac{\pi^2\cos\pi\alpha}{\sin^2\pi\alpha};$$

Положим

$$\alpha = \frac{2}{3}:\quad \int\limits_{-1}^{1}\ln\frac{1+x}{1-x}\frac{dx}{\sqrt[3]{(1-x)^2(1+x)}} = \frac{2\pi^2}{3}$$

2. $I = \int\limits_2^{+\infty}\frac{\ln(x-2)}{(x^2-1)\sqrt{x-2}}dx = -\frac{\pi\ln 3}{2\sqrt{3}}$

Замечая, что $\frac{1}{x^2-1} = \frac{1}{2}\left(\frac{1}{x-1} - \frac{1}{x+1}\right)$, запишем интеграл в виде $I = \frac{1}{2}(I_1 - I_2)$, где

$$I_1=\int_2^{+\infty}\frac{\ln(x-2)}{(x-1)\sqrt{x-2}}\,dx,\ \ I_2=\int_2^{+\infty}\frac{\ln(x-2)}{(x+1)\sqrt{x-2}}\,dx$$

Для I_1 можем написать

$$I_1=\int_2^{+\infty}\frac{\ln(x-2)}{(x-1)\sqrt{x-2}}\,dx=\int_0^{+\infty}\frac{\ln y}{(y+1)\sqrt{y}}\,dy=\left(\frac{d}{d\alpha}\int_0^{+\infty}\frac{y^{\alpha-1}}{(y+1)}\,dy\right)\Bigg|_{\alpha=1/2}=$$

$$=\left(\frac{d}{d\alpha}\frac{\pi}{\sin\pi\alpha}\right)\Bigg|_{\alpha=1/2}=0$$

А для I_2 —

$$I_2=\int_2^{+\infty}\frac{\ln(x-2)}{(x+1)\sqrt{x-2}}\,dx=\int_0^{+\infty}\frac{\ln z}{(z+3)\sqrt{z}}\,dz=\left[z=3y\right]=\frac{1}{\sqrt{3}}\int_0^{+\infty}\frac{\ln 3y}{(y+1)\sqrt{y}}\,dy=$$

$$=\frac{1}{\sqrt{3}}\left(\int_0^{+\infty}\frac{\ln 3}{(y+1)\sqrt{y}}\,dy+\int_0^{+\infty}\frac{\ln y}{(y+1)\sqrt{y}}\,dy\right)=\frac{\ln 3}{\sqrt{3}}\int_0^{+\infty}\frac{1}{(y+1)\sqrt{y}}\,dy=\frac{\ln 3}{\sqrt{3}}\cdot\frac{\pi}{\sin\frac{\pi}{2}}=\frac{\pi\ln 3}{\sqrt{3}}$$

3. $$\int_0^{\pi/2}\frac{\operatorname{tg}^{\alpha}x\,dx}{(\sin x+\cos x)^2}\,dx=\frac{\pi\alpha}{\sin\pi\alpha},\ \ 0<\alpha<1$$

$$\int_0^{\pi/2}\frac{\operatorname{tg}^{\alpha}x\,dx}{(\sin x+\cos x)^2}\,dx=\int_0^{\pi/2}\frac{\operatorname{tg}^{\alpha}x\,dx}{(\operatorname{tg}x+1)^2}\frac{dx}{\cos^2 x}=\int_0^{+\infty}\frac{z^{\alpha}dz}{(z+1)^2}=\mathrm{B}(1+\alpha,1-\alpha)=$$

$$=\frac{\Gamma(1+\alpha)\Gamma(1-\alpha)}{\Gamma(2)}=\alpha\Gamma(\alpha)\Gamma(1-\alpha)=\frac{\pi\alpha}{\sin\pi\alpha}$$

4. $$\int_0^1\frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx=\mathrm{B}(p,q)$$

$$\int_0^1\frac{x^{q-1}}{(1+x)^{p+q}}\,dx=\left[x=\frac{1}{y}\right]=\int_1^{+\infty}\frac{\left(\frac{1}{y}\right)^{q-1}}{\left(1+\frac{1}{y}\right)^{p+q}}\frac{dy}{y^2}=\int_1^{+\infty}\frac{y^{p-1}}{(1+y)^{p+q}}\,dy;$$

$$\int_0^1\frac{x^{p-1}+x^{q-1}}{(1+x)^{p+q}}\,dx=\int_0^{+\infty}\frac{y^{p-1}}{(1+y)^{p+q}}\,dy=\mathrm{B}(p,\,q)$$

5. $$\int_0^1\frac{dx}{\sqrt{1-x^4}}\cdot\int_0^1\frac{x^2dx}{\sqrt{1-x^4}}=\frac{\pi}{4}$$

$$\int_0^1 \frac{dx}{\sqrt{1-x^4}} \cdot \int_0^1 \frac{x^2 dx}{\sqrt{1-x^4}} = \left[y = x^4 \right] = \frac{1}{16} \int_0^1 \frac{y^{-3/4} dy}{\sqrt{1-y}} \cdot \int_0^1 \frac{y^{-1/4} dy}{\sqrt{1-y}} =$$

$$= \frac{1}{16} \mathrm{B}\left(\frac{1}{4}, \frac{1}{2} \right) \cdot \mathrm{B}\left(\frac{3}{4}, \frac{1}{2} \right) = \frac{1}{16} \cdot \frac{\Gamma\left(\frac{1}{4} \right) \Gamma\left(\frac{1}{2} \right)}{\Gamma\left(\frac{3}{4} \right)} \cdot \frac{\Gamma\left(\frac{3}{4} \right) \Gamma\left(\frac{1}{2} \right)}{\Gamma\left(\frac{5}{4} \right)} = \frac{\pi}{4}$$

6. $\displaystyle\int_0^{+\infty} e^{-x^4} dx \cdot \int_0^{+\infty} x^2 e^{-x^4} dx = \frac{\pi}{8\sqrt{2}}$

7. $\displaystyle\mathrm{B}(p, p) = \frac{1}{2^{2p-1}} \mathrm{B}\left(\frac{1}{2}, p \right)$

Интеграл $\displaystyle I = \int_{-1}^1 \left(1 - x^2\right)^{p-1} dx$ вычислим двумя способами.

С одной стороны, $\displaystyle I = \int_{-1}^1 \left(1 - x^2\right)^{p-1} dx = \left[x = 2y - 1 \right] = 2^{2p-1} \int_0^1 \left(y(1-y) \right)^{p-1} dy = 2^{2p-1} \mathrm{B}(p, p)$.

С другой стороны,
$\displaystyle I = \int_{-1}^1 \left(1 - x^2\right)^{p-1} dx = 2\int_0^1 \left(1 - x^2\right)^{p-1} dx = \left[z = x^2 \right] = \int_0^1 (1-z)^{p-1} z^{-1/2} dy = \mathrm{B}\left(p, \frac{1}{2} \right)$, и мы приходим к требуемому равенству.

8. $\displaystyle I = \int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx = \pi \operatorname{ctg} \pi p$

$$I = \int_0^1 \frac{x^{p-1} - x^{-p}}{1-x} dx = \lim_{\varepsilon \to +0} \int_0^1 \frac{x^{p-1} - x^{-p}}{(1-x)^{1-\varepsilon}} dx = \lim_{\varepsilon \to +0} \left(\mathrm{B}(p, \varepsilon) - \mathrm{B}(1-p, \varepsilon) \right) =$$

$$= \lim_{\varepsilon \to +0} \Gamma(\varepsilon) \left(\frac{\Gamma(p)}{\Gamma(p+\varepsilon)} - \frac{\Gamma(1-p)}{\Gamma(1-p+\varepsilon)} \right) = \lim_{\varepsilon \to +0} \frac{1}{\varepsilon} \left(\frac{\Gamma(p)}{\Gamma(p+\varepsilon)} - \frac{\Gamma(1-p)}{\Gamma(1-p+\varepsilon)} \right) =$$

$$= \lim_{\varepsilon \to +0} \frac{1}{\Gamma(p)\Gamma(1-p)} \frac{\Gamma(p)\Gamma(1-p+\varepsilon) - \Gamma(1-p)\Gamma(p+\varepsilon)}{\varepsilon} = \frac{\Gamma(p)\Gamma'(1-p) - \Gamma'(p)\Gamma(1-p)}{\Gamma(p)\Gamma(1-p)} =$$

$$= -\frac{\left(\Gamma(p)\Gamma(1-p) \right)'}{\Gamma(p)\Gamma(1-p)} = -\frac{\left(\dfrac{\pi}{\sin \pi p} \right)'}{\dfrac{\pi}{\sin \pi p}} = \frac{\pi \cos \pi p}{\sin \pi p} = \pi \operatorname{ctg} \pi p$$

9. $\displaystyle\int_0^{+\infty} \frac{\operatorname{sh} \alpha x}{\operatorname{sh} \beta x} dx = \frac{\pi}{2\beta} \operatorname{tg} \frac{\pi \alpha}{2\beta}$

$$\int_0^{+\infty} \frac{\operatorname{sh} \alpha x}{\operatorname{sh} \beta x} dx = \int_0^{+\infty} \frac{e^{\alpha x} - e^{-\alpha x}}{e^{\beta x} - e^{-\beta x}} dx = \int_0^{+\infty} \frac{e^{(\beta+\alpha)x} - e^{(\beta-\alpha)x}}{1 - e^{-2\beta x}} e^{-2\beta x} dx = \left[u = e^{-2\beta x} \right] =$$

$$= \frac{1}{2\beta} \int_0^1 \frac{u^{-\frac{\beta+\alpha}{2\beta}} - u^{\frac{\alpha-\beta}{2\beta}}}{1-u} du = \frac{\pi}{2\beta} \operatorname{ctg} \pi \frac{\beta - \alpha}{2\beta} = \frac{\pi}{2\beta} \operatorname{tg} \frac{\pi \alpha}{2\beta}$$