КОНТРОЛЬНАЯ РАБОТА "ИНТЕГРАЛЫ, ЗАВИСЯЩИЕ ОТ ПАРАМЕТРА"

Вычислите интегралы.

1. $\int\limits_0^{+\infty} \frac{\operatorname{arctg} x}{x\left(x^2+4\right)} dx$

$$I\left(a\right)=\int\limits_0^{+\infty} \frac{\operatorname{arctg} \alpha x}{x\left(x^2+4\right)} dx$$

$$I'\left(a\right)=\int\limits_0^{+\infty} \frac{1}{\left(1+\alpha^2x^2\right)\left(x^2+4\right)} dx=\frac{1}{4\alpha^2-1}\int\limits_0^{+\infty}\left(\frac{\alpha^2}{1+\alpha^2x^2}-\frac{1}{x^2+4}\right)dx=$$

$$=\frac{\pi}{2}\frac{\alpha-\frac{1}{2}}{4\alpha^2-1}=\frac{\pi}{2\left(2\alpha+1\right)2}=\frac{\pi}{8\left(\alpha+\frac{1}{2}\right)}$$

$$I\left(\alpha\right)=\frac{\pi}{8}\left(\ln\left(\alpha+\frac{1}{2}\right)+C\right)$$

$$0=I\left(0\right)=\frac{\pi}{8}\left(\ln\left(\frac{1}{2}\right)+C\right), C=\ln 2$$

$$I\left(\alpha\right)=\frac{\pi}{8}\ln\left(2\alpha+1\right),$$

$$\int\limits_0^{+\infty} \frac{\operatorname{arctg} x}{x\left(x^2+4\right)} dx=\frac{\pi}{8}\ln 3$$

2. $\int\limits_0^{+\infty} \frac{e^{-4x^2}-e^{-9x^2}}{x} dx=\ln\frac{3}{2}$

3. $$\int\limits_0^{+\infty} \frac{\sin^2 x\cos 2x}{x^2} dx=-\frac{1}{x}\sin^2 x\cos 2x\Big|_0^{+\infty}+\int\limits_0^{+\infty}\frac{2\sin x\cos x\cos 2x-2\sin^2 x\sin 2x}{x}dx=$$

$$=\int\limits_0^{+\infty}\frac{\frac{1}{2}\sin 4x-\left(1-\cos 2x\right)\sin 2x}{x}dx=\int\limits_0^{+\infty}\frac{\frac{1}{2}\sin 4x-\sin 2x+\frac{1}{2}\sin 4x}{x}dx=0$$

4. $\int\limits_0^{+\infty} \frac{\ln x}{\left(x^2+1\right)^2} dx$

$$I(\alpha)=\int_0^{+\infty}\frac{x^\alpha}{\left(x^2+1\right)^2}dx=\left[y=x^2,x=y^{1/2}\right]=\frac{1}{2}\int_0^{+\infty}\frac{y^{\frac{\alpha}{2}-\frac{1}{2}}}{\left(y+1\right)^2}dy=$$

$$=\frac{1}{2}\mathrm{B}\left(\frac{\alpha}{2}+\frac{1}{2},-\frac{\alpha}{2}+\frac{3}{2}\right)=\frac{1}{2}\frac{\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\alpha}{2}+\frac{3}{2}\right)}{\Gamma(2)}=$$

$$=\frac{1}{2}\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\alpha}{2}+\frac{3}{2}\right)=\frac{1}{2}\Gamma\left(\frac{\alpha}{2}+\frac{1}{2}\right)\Gamma\left(-\frac{\alpha}{2}+\frac{1}{2}\right)\left(-\frac{\alpha}{2}+\frac{1}{2}\right)=$$

$$=\frac{1}{2}\frac{\pi}{\sin\pi\left(\frac{1}{2}-\frac{\alpha}{2}\right)}\left(-\frac{\alpha}{2}+\frac{1}{2}\right)=\frac{1}{2}\frac{\pi}{\cos\frac{\pi\alpha}{2}}\left(-\frac{\alpha}{2}+\frac{1}{2}\right)$$

$$\int_0^{+\infty}\frac{x^\alpha}{\left(x^2+1\right)^2}dx=\frac{1}{2}\frac{\pi}{\cos\frac{\pi\alpha}{2}}\left(-\frac{\alpha}{2}+\frac{1}{2}\right);$$

$$\int_0^{+\infty}\frac{x^\alpha\ln x}{\left(x^2+1\right)^2}dx=\frac{1}{4}\frac{\pi^2\sin\frac{\pi\alpha}{2}}{\cos^2\frac{\pi\alpha}{2}}\left(-\frac{\alpha}{2}+\frac{1}{2}\right)-\frac{1}{4}\frac{\pi}{\cos\frac{\pi\alpha}{2}}$$

$$\alpha=0:\int_0^{+\infty}\frac{\ln x}{\left(x^2+1\right)^2}dx=-\frac{\pi}{4}$$