ТРОЙНОЙ ИНТЕГРАЛ

4077. $\iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3}$, тело V ограничено поверхностями $x+y+z=1, x=0, y=0, z=0.$

$$\iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3} = \int_0^1 dx \int_0^{1-x} dy \int_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} =$$

$$= \int_0^1 dx \int_0^{1-x} dy \left(-\frac{1}{2(1+x+y+z)^2} \Bigg|_{z=0}^{1-x-y} \right) = \frac{1}{2}\int_0^1 dx \int_0^{1-x} \left(\frac{1}{(1+x+y)^2} - \frac{1}{4} \right) dy =$$

$$= \frac{1}{2}\int_0^1 dx \int_0^{1-x} \frac{1}{(1+x+y)^2} dy - \frac{1}{16} = \frac{1}{2}\int_0^1 dx \left(-\frac{1}{(1+x+y)} \Bigg|_{y=0}^{1-x} \right) - \frac{1}{16} =$$

$$= \frac{1}{2}\int_0^1 \left(\frac{1}{1+x} - \frac{1}{2} \right) dx - \frac{1}{16} = \frac{1}{2}\ln(1+x)\Big|_0^1 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2}\ln 2 - \frac{5}{16}.$$

4080. $\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz$, тело V ограничено поверхностями $x^2+y^2=z^2, z=1.$

$$\iiint\limits_V \sqrt{x^2+y^2}\,dxdydz = \int_0^1 dz \iint\limits_{x^2+y^2\le z^2} \sqrt{x^2+y^2}\,dxdy = 2\pi\int_0^1 dz \int_0^z r^2 dr =$$

$$= \frac{2\pi}{3}\int_0^1 z^3 dz = \frac{2\pi}{3}\frac{1}{4} = \frac{\pi}{6}.$$

4088.

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz = \iint\limits_{\substack{x^2+y^2\le 1,\\ x,y\ge 0}} dxdy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz =$$

$$= \frac{1}{3}\iint\limits_{\substack{x^2+y^2\le 1,\\ x,y\ge 0}} \left(\left(2-x^2-y^2\right)^{3/2} - \left(x^2+y^2\right)^{3/2} \right) dxdy = \begin{bmatrix} x = r\cos\varphi, \\ y = r\sin\varphi \end{bmatrix} =$$

$$= \frac{1}{3}\iint\limits_{\substack{0\le r\le 1\\ 0\le\varphi\le\pi/2}} \left(\left(2-r^2\right)^{3/2} - r^3 \right) rdrd\varphi = \frac{\pi}{6}\int_0^1 \left(\left(2-r^2\right)^{3/2} - r^3 \right) rdr =$$

$$\frac{\pi}{6}\left(-\frac{1}{5}\left(2-r^2\right)^{5/2} - \frac{1}{5}r^5 \right)\Bigg|_0^1 = \frac{\pi}{6}\left(\frac{1}{5}\left(2^{5/2}-1\right) - \frac{1}{5} \right) = \frac{\pi}{6}\left(\frac{2^{5/2}}{5} - \frac{2}{5} \right) = \frac{\pi}{15}\left(2\sqrt{2}-1\right)$$

или

$$\int_0^1 dx \int_0^{\sqrt{1-x^2}} dy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz = \iint_{\substack{x^2+y^2\le 1,\\ x,y\ge 0}} dxdy \int_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}} z^2 dz =$$

$$= \begin{bmatrix} x=\rho\cos\varphi\cos\psi \\ y=\rho\sin\varphi\cos\psi \\ z=\rho\sin\psi \end{bmatrix} = \iiint_{\substack{0\le\varphi\le\frac{\pi}{2}\\ \frac{\pi}{4}\le\psi\le\frac{\pi}{2}\\ 0\le\rho\le\sqrt{2}}} \rho^2\sin^2\psi\rho^2\cos\psi\, d\rho d\varphi d\psi =$$

$$= \frac{\pi}{2}\int_0^{\sqrt{2}} \rho^4 d\rho \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin^2\psi\cos\psi\, d\psi = \frac{\pi}{2}\frac{2^{5/2}}{5}\frac{1-\frac{1}{2^{3/2}}}{3} = \frac{\pi}{15}\left(2\sqrt{2}-1\right).$$

4092. $\iiint\limits_V x^2 dxdydz$, тело V ограничено поверхностями $z=ay^2, z=by^2, y>0$ $\left(0<a<b\right)$, $z=\alpha x, z=\beta x$ $\left(0<\alpha<\beta\right)$, $z=h$ $\left(h>0\right)$.

$$\iiint\limits_V x^2 dxdydz = \int_0^h dz \iint_{\substack{\frac{z}{\beta}\le x\le\frac{z}{\alpha}\\ \sqrt{\frac{z}{b}}\le y\le\sqrt{\frac{z}{a}}}} x^2 dxdy = \int_0^h dz \int_{\frac{z}{\beta}}^{\frac{z}{\alpha}} x^2 dx \int_{\sqrt{\frac{z}{b}}}^{\sqrt{\frac{z}{a}}} dy =$$

$$= \frac{1}{3}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)\left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}\right)\int_0^h z^3\sqrt{z}dz = \frac{2}{27}h^4\sqrt{h}\left(\frac{1}{\alpha^3}-\frac{1}{\beta^3}\right)\left(\frac{1}{\sqrt{a}}-\frac{1}{\sqrt{b}}\right).$$

Найдите объемы тел, ограниченных следующими поверхностями:

4102. $z=x+y, z=xy, x+y=1, x=0, y=0.$

$$V = \iint_{\substack{x,y\ge 0\\ x+y\le 1}} \left(x+y-xy\right)dxdy = \frac{1}{3}-\int_0^1 xdx\int_0^{1-x} ydy = \frac{1}{3}-\frac{1}{2}\int_0^1 x\left(1-x\right)^2 dx = \frac{1}{3}-\frac{1}{6}\cdot\frac{1}{4} = \frac{1}{3}-\frac{1}{24} = \frac{7}{24}.$$

4108. $\left(x^2+y^2+z^2\right)^2 = a^2\left(x^2+y^2-z^2\right).$

$$V=\iiint\limits_V dxdydz=\iiint\limits_{\substack{V_1:0\le\varphi\le 2\pi\\ -\frac{\pi}{4}\le\psi\le\frac{\pi}{4}\\ \rho^2\le a^2\cos 2\psi}}\rho^2\cos\psi\, d\rho d\varphi d\psi=$$

$$2\pi\cdot 2\int_0^{\pi/4}\cos\psi\, d\psi\int_0^{a\sqrt{\cos 2\psi}}\rho^2 d\rho=\frac{4\pi a^3}{3}\int_0^{\pi/4}\cos\psi\cos^{3/2}2\psi\, d\psi=$$

$$=\frac{4\pi a^3}{3}\int_0^{1/\sqrt{2}}\left(1-2u^2\right)^{3/2}du=\frac{4\pi a^3}{3\sqrt{2}}\int_0^1\left(1-v^2\right)^{3/2}dv=$$

$$=\frac{4\pi a^3}{3\sqrt{2}}\int_0^{\pi/2}\cos^4 t\,dt=\frac{4\pi a^3}{3\sqrt{2}}\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2}=\frac{\pi^2 a^3}{4\sqrt{2}}.$$

4118 б). $\sqrt[3]{\frac{x}{a}}+\sqrt[3]{\frac{y}{b}}+\sqrt[3]{\frac{z}{c}}=1,\ \ a,b,c>0;\ x,y,z\ge 0.$

$$V=\iiint\limits_{\substack{x,y,z\ge 0\\ \sqrt[3]{\frac{x}{a}}+\sqrt[3]{\frac{y}{b}}+\sqrt[3]{\frac{z}{c}}\le 1}}dxdydz=\begin{bmatrix}x=au\\ y=bv\\ z=cw\end{bmatrix}=abc\iiint\limits_{\substack{u,v,w\ge 0\\ \sqrt[3]{u}+\sqrt[3]{v}+\sqrt[3]{w}\le 1}}dxdydz=\begin{bmatrix}u=\xi^3\\ v=\eta^3\\ w=\zeta^3\end{bmatrix}=$$

$$=27abc\iiint\limits_{\substack{\xi,\eta,\zeta\ge 0\\ \zeta+\eta+\zeta\le 1}}\xi^2\eta^2\zeta^2 d\xi d\eta d\zeta=27abc\int_0^1\xi^2 d\xi\int_0^{1-\xi}\eta^2 d\eta\int_0^{1-\xi-\eta}\zeta^2 d\zeta=$$

$$=9abc\int_0^1\xi^2 d\xi\int_0^{1-\xi}\eta^2\left(1-\xi-\eta\right)^3 d\eta=9abc\frac{3}{3}\frac{2}{4}\frac{1}{5}\frac{1}{6}\int_0^1\xi^2\left(1-\xi\right)^6 d\xi=$$

$$=\frac{3}{20}abc\int_0^1\xi^2\left(1-\xi\right)^6 d\xi=\frac{3}{20}abc\frac{2}{7}\frac{1}{8}\frac{1}{9}=\frac{1}{20}abc\frac{2}{7}\frac{1}{8}\frac{1}{3}=\frac{abc}{1680}.$$