Занятие 8

Дифференциальные уравнения. не разрешенные относительно производной.

244. $y^2\left(y'^2+1\right)=1$.

$$y'^2=\frac{1}{y^2}-1,\ y'^2=\frac{1-y^2}{y^2},\ y'=\pm\frac{\sqrt{1-y^2}}{y},$$

$$\frac{yy'}{\sqrt{1-y^2}}=\pm 1,\ \sqrt{1-y^2}=\pm\left(x+C\right),\ \left(x+C\right)^2+y^2=1$$

Ответ: $\left(x+C\right)^2+y^2=1, y=\pm 1$

247. $xy'^2=y, \quad x,y>0$.

$$y'=\pm\sqrt{\frac{y}{x}},\ \frac{y'}{\sqrt{y}}=\pm\frac{1}{\sqrt{x}},\ \sqrt{y}=\pm\sqrt{x}+C$$

Ответ: $\sqrt{y}=\pm\sqrt{x}+C,\ y=0$,

253. $xy'^2-2yy'+x=0$.

$$y'=\frac{y\pm\sqrt{y^2-x^2}}{x},\ y'=\frac{y}{x}\pm\sqrt{\frac{y^2}{x^2}-1},$$

$$z=\frac{y}{x}, y=xz,$$

$$xz'=\pm\sqrt{z^2-1},\ \pm\frac{z'}{\sqrt{z^2-1}}+\frac{1}{x}=0,$$

$$\pm\ln\left|z+\sqrt{z^2-1}\right|+\ln|x|=0,$$

$$\ln\left|z\pm\sqrt{z^2-1}\right|+\ln|x|=\ln C,$$

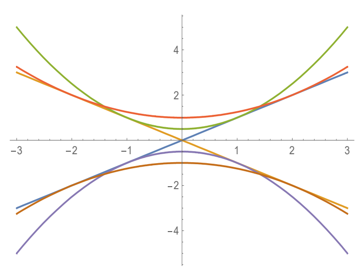
$$zx\pm\sqrt{z^2x^2-x^2}=C,$$

$$y\pm\sqrt{y^2-x^2}=C,$$

$$\pm\sqrt{y^2-x^2}=C-y,$$

$$y^2-x^2=C^2-2Cy+y^2,$$

$$2Cy=C^2+x^2$$



Ответ: $2Cy=C^2+x^2,\ y=\pm x$.

258. $\left(xy'+3y\right)^2=7x$.

$$xy'+3y=\pm\sqrt{7x}$$
$$x^3y'+3x^2=\pm\sqrt{7}x^{5/2}$$
$$x^3y=\pm\frac{2}{\sqrt{7}}x^{7/2}+C$$
$$y=\pm\frac{2}{\sqrt{7}}\sqrt{x}+\frac{C}{x^3}$$

262. $x\left(y-xy'\right)^2=xy'^2-2yy'$.

$$x\left(y-xy'\right)^2=xy'^2-2yy'$$
$$y=xz$$
$$x\left(xz-xz-x^2z'\right)^2=x\left(z+xz'\right)^2-2xz\left(z+xz'\right)$$
$$x^4z'^2=\left(z+xz'\right)^2-2z\left(z+xz'\right)$$
$$x^4z'^2=z^2+2xzz'+x^2z'^2-2z^2-2xzz'$$
$$x^4z'^2=x^2z'^2-z^2$$
$$\left(x^4-x^2\right)z'^2=-z^2$$
$$x^2\left(1-x^2\right)z'^2=z^2$$
$$x\sqrt{1-x^2}z'=\pm z$$
$$\frac{z'}{z}=\pm\frac{1}{x\sqrt{1-x^2}}$$
$$\ln Cz=\pm\ln\left|\frac{1}{x}+\sqrt{\frac{1}{x^2}-1}\right|$$
$$Cz=\frac{1}{x}\pm\sqrt{\frac{1}{x^2}-1}$$
$$Cy=1\pm\sqrt{1-x^2}$$
$$Cy-1=\pm\sqrt{1-x^2}$$
$$\left(Cy-1\right)^2=1-x^2$$
$$x^2+\left(Cy-1\right)^2=1$$

Ответ: $x^2+\left(Cy-1\right)^2=1, y=0$

или

$$x\left(y-xy'\right)^2=xy'^2-2yy'$$
$$xy^2-2x^2yy'+x^3y'^2=xy'^2-2yy'$$
$$\left(x^3-x\right)y'^2-2\left(x^2-1\right)yy'+xy^2=0$$
$$y'=\frac{\left(x^2-1\right)y\pm\sqrt{\left(x^2-1\right)^2y^2-xy^2\left(x^3-x\right)}}{\left(x^3-x\right)}$$
$$y'=y\frac{\left(x^2-1\right)\pm\sqrt{1-x^2}}{x\left(x^2-1\right)}$$
$$y'=y\left(\frac{1}{x}\pm\frac{1}{x\sqrt{1-x^2}}\right)$$
$$\frac{y'}{y}=\frac{1}{x}\pm\frac{1}{x\sqrt{1-x^2}}$$
$$\ln Cy=\ln x\pm\ln\frac{1+\sqrt{1-x^2}}{x}$$
$$\ln Cy=\ln x+\ln\frac{1\pm\sqrt{1-x^2}}{x}$$
$$Cy=1\pm\sqrt{1-x^2}$$
$$Cy-1=\sqrt{1-x^2}$$
$$\left(Cy-1\right)^2=1-x^2$$
$$x^2+\left(Cy-1\right)^2=1$$

Решите дифференциальные уравнения методом введения параметра.

267. $x=y'^3+y'$.

$$x=p^3+p$$
$$dx=3p^2dp+dp$$
$$pdx=dy$$
$$\left(3p^3+p\right)dp=dy$$

Ответ: $\begin{cases}x=p^3+p\\ y=p^4+\dfrac{p^2}{2}+C\end{cases}$

272. $y=\ln\left(1+y'^2\right)$.

$$y=\ln\left(1+p^2\right)$$

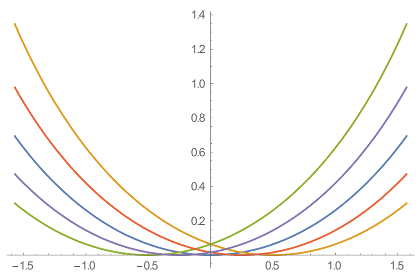
$$dy=\frac{2p}{1+p^2}dp$$

$$pdx=\frac{2p}{1+p^2}dp$$

$$dx=\frac{2}{1+p^2}dp$$

$$x+C=2\,\mathrm{arctg}\,p$$

$$p=\mathrm{tg}\frac{x+C}{2}$$



$$y=\ln\left(1+\mathrm{tg}^2\frac{x+C}{2}\right)$$

$$y=\ln\frac{1}{\cos^2\frac{x+C}{2}}$$

$$y=-2\ln\cos\frac{x+C}{2}$$

Ответ: $y=-2\ln\cos\dfrac{x+C}{2},\ y=0$

284. $y=xy'-x^2y'^3$.

$$y=xp-x^2p^3$$

$$dy=pdx+xdp-2xp^3dx-3x^2p^2dp$$

$$pdx=pdx+xdp-2xp^3dx-3x^2p^2dp$$

$$xdp-2xp^3dx-3x^2p^2dp=0$$

$$dp-2p^3dx-3xp^2dp=0$$

$$2p^3dx+3xp^2dp=dp$$

$$2p^{3/2}dx+3xp^{1/2}dp=\frac{dp}{p^{3/2}}$$

$$2xp^{3/2}=-\frac{2}{p^{1/2}}+C$$

$$x=-\frac{1}{p^2}+\frac{C}{p^{3/2}}$$

$$\begin{cases} x = -\dfrac{1}{p^2} + \dfrac{C}{p^{3/2}}, \\ y = xp - x^2 p^3, \end{cases} \quad y = 0$$

Решите уравнения Лагранжа и Клеро.

$y = 2xy' - y'^2$.

$$y = 2xp - p^2$$
$$dy = 2pdx + 2xdp - 2pdp$$
$$pdx = 2pdx + 2xdp - 2pdp$$
$$pdx + 2xdp - 2pdp = 0$$
$$p^2dx + 2xpdp - 2p^2dp = 0$$
$$xp^2 = \frac{2}{3}p^3 + C$$
$$x = \frac{2}{3}p + \frac{C}{p^2}$$
$$y = 2p\left(\frac{2}{3}p + \frac{C}{p^2}\right) - p^2$$

Ответ: $\begin{cases} x = \dfrac{2}{3}p + \dfrac{C}{p^2}, \\ y = \dfrac{p^2}{3} + \dfrac{C}{3}, \end{cases} \quad y = 0$

289. $y = 2xy' - 4y'^3$.

$$y = 2xp - 4p^3$$
$$dy = 2pdx + 2xdp - 12p^2dp$$
$$pdx = 2pdx + 2xdp - 12p^2dp$$
$$pdx + 2xdp - 12p^2dp = 0$$
$$p^2dx + 2xpdp - 12p^3dp = 0$$
$$xp^2 = 3p^4 + C$$
$$x = 3p^2 + \frac{C}{p^2}$$
$$y = 2p\left(3p^2 + \frac{C}{p^2}\right) - 4p^3$$

Ответ: $\begin{cases} x = 3p^2 + \dfrac{C}{p^2} \\ y = 2p^3 + \dfrac{2C}{p} \end{cases}$ $\qquad y = 0$

293. $xy' - y = \ln y'$.

$$y = xy' - \ln y'$$
$$y = xp - \ln p$$
$$dy = pdx + xdp - \frac{dp}{p}$$
$$pdx = pdx + xdp - \frac{dp}{p}$$
$$\left(x - \frac{1}{p}\right)dp = 0$$
$$dp = 0, p = C,\ y = Cx - \ln C$$
$$x = \frac{1}{p}, p = \frac{1}{x},\ y = 1 + \ln x$$

Ответ: $y = Cx - \ln C,\ y = 1 + \ln x$