# ТРОЙНОЙ ИНТЕГРАЛ

**4077.** $\iiint\limits_V \dfrac{dxdydz}{(1+x+y+z)^3}$ , тело $V$ ограничено поверхностями $x+y+z=1, x=0, y=0, z=0$.

$$\iiint\limits_V \frac{dxdydz}{(1+x+y+z)^3} = \int\limits_0^1 dx \int\limits_0^{1-x} dy \int\limits_0^{1-x-y} \frac{dz}{(1+x+y+z)^3} =$$

$$= \int\limits_0^1 dx \int\limits_0^{1-x} dy \left( -\frac{1}{2(1+x+y+z)^2}\bigg|_{z=0}^{1-x-y} \right) = \frac{1}{2}\int\limits_0^1 dx \int\limits_0^{1-x} \left( \frac{1}{(1+x+y)^2} - \frac{1}{4} \right) dy =$$

$$= \frac{1}{2}\int\limits_0^1 dx \int\limits_0^{1-x} \frac{1}{(1+x+y)^2}\, dy - \frac{1}{16} = \frac{1}{2}\int\limits_0^1 dx \left( -\frac{1}{(1+x+y)}\bigg|_{y=0}^{1-x} \right) - \frac{1}{16} =$$

$$= \frac{1}{2}\int\limits_0^1 \left( \frac{1}{1+x} - \frac{1}{2} \right) dx - \frac{1}{16} = \frac{1}{2}\ln(1+x)\bigg|_0^1 - \frac{1}{4} - \frac{1}{16} = \frac{1}{2}\ln 2 - \frac{5}{16}.$$

**4079.** $\iiint\limits_V \left( \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} \right) dxdydz$ , тело $V$ ограничено поверхностью $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$ .

$$\iiint\limits_V \left( \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \right) dxdydz = [x=au, y=bv, z=cw] = abc \iiint\limits_{V_1: u^2+v^2+w^2 \le 1} \left( u^2 + v^2 + w^2 \right) dxdydz =$$

$$= 3abc\iiint\limits_{V_1} w^2 = 6abc\int\limits_0^1 w^2 dw \iint\limits_{u^2+v^2 \le 1-w^2} dudv = 6\pi abc\int\limits_0^1 w^2 \left( 1-w^2 \right) dw = 6\pi abc \left( \frac{1}{3} - \frac{1}{5} \right) = \frac{4}{5}\pi abc$$

ИЛИ

$$\iiint\limits_{V_1: u^2+v^2+w^2 \le 1} \left( u^2 + v^2 + w^2 \right) dxdydz = [x=\rho\cos\varphi\cos\psi, y=\rho\sin\varphi\cos\psi, z=\rho\sin\psi]$$

$$= \iiint\limits_{\substack{0 \le \rho \le 1 \\ 0 \le \varphi \le 2\pi \\ -\frac{\pi}{2} \le \psi \le \frac{\pi}{2}}} \rho^4 \cos\psi\, d\rho d\varphi d\psi = \int\limits_0^1 \rho^4 d\rho \int\limits_0^{2\pi} d\varphi \int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\psi\, d\psi = \frac{1}{5}\cdot 2\pi \cdot 2 = \frac{4}{5}\pi$$

**4080.** $\iiint\limits_V \sqrt{x^2 + y^2}\, dxdydz$ , тело $V$ ограничено поверхностями $x^2 + y^2 = z^2, z = 1$.

$$\iiint\limits_{V} \sqrt{x^2+y^2}\,dxdydz = \int\limits_{0}^{1}dz \iint\limits_{x^2+y^2\leq z^2} \sqrt{x^2+y^2}\,dxdy = 2\pi\int\limits_{0}^{1}dz\int\limits_{0}^{z}r^2dr =$$

$$= \frac{2\pi}{3}\int\limits_{0}^{1}z^3dz = \frac{2\pi}{3}\frac{1}{4} = \frac{\pi}{6}.$$

4088.

$$\int\limits_{0}^{1}dx\int\limits_{0}^{\sqrt{1-x^2}}dy\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}z^2dz = \iint\limits_{\substack{x^2+y^2\leq 1,\\ x,y\geq 0}}dxdy\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}z^2dz =$$

$$= \frac{1}{3}\iint\limits_{\substack{x^2+y^2\leq 1,\\ x,y\geq 0}}\left(\left(2-x^2-y^2\right)^{3/2}-\left(x^2+y^2\right)^{3/2}\right)dxdy = \begin{bmatrix} x=r\cos\varphi,\\ y=r\sin\varphi \end{bmatrix} =$$

.

$$= \frac{1}{3}\iint\limits_{\substack{0\leq r\leq 1\\ 0\leq\varphi\leq\pi/2}}\left(\left(2-r^2\right)^{3/2}-r^3\right)rdrd\varphi = \frac{\pi}{6}\int\limits_{0}^{1}\left(\left(2-r^2\right)^{3/2}-r^3\right)rdr =$$

$$\frac{\pi}{6}\left(-\frac{1}{5}\left(2-r^2\right)^{5/2}-\frac{1}{5}r^5\right)\Bigg|_{0}^{1} = \frac{\pi}{6}\left(\frac{1}{5}\left(2^{5/2}-1\right)-\frac{1}{5}\right) = \frac{\pi}{6}\left(\frac{2^{5/2}}{5}-\frac{2}{5}\right) = \frac{\pi}{15}\left(2\sqrt{2}-1\right)$$

или

$$\int\limits_{0}^{1}dx\int\limits_{0}^{\sqrt{1-x^2}}dy\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}z^2dz = \iint\limits_{\substack{x^2+y^2\leq 1,\\ x,y\geq 0}}dxdy\int\limits_{\sqrt{x^2+y^2}}^{\sqrt{2-x^2-y^2}}z^2dz =$$

$$= \begin{bmatrix} x=\rho\cos\varphi\cos\psi\\ y=\rho\sin\varphi\cos\psi\\ z=\rho\sin\psi \end{bmatrix} = \iiint\limits_{\substack{0\leq\varphi\leq\frac{\pi}{2}\\ \frac{\pi}{4}\leq\psi\leq\frac{\pi}{2}\\ 0\leq\rho\leq\sqrt{2}}}\rho^2\sin^2\psi\rho^2\cos\psi\,d\rho d\varphi d\psi =$$

$$= \frac{\pi}{2}\int\limits_{0}^{\sqrt{2}}\rho^4d\rho\int\limits_{\frac{\pi}{4}}^{\frac{\pi}{2}}\sin^2\psi\cos\psi\,d\psi = \frac{\pi}{2}\frac{2^{5/2}}{5}\frac{1-\dfrac{1}{2^{3/2}}}{3} = \frac{\pi}{15}\left(2\sqrt{2}-1\right).$$

4092. $\iiint\limits_{V}x^2dxdydz$, тело $V$ ограничено поверхностями $z=ay^2, z=by^2, y>0\ \left(0<a<b\right),$

$z=\alpha x, z=\beta x\ \left(0<\alpha<\beta\right), z=h\ \left(h>0\right).$

$$\iiint\limits_{V} x^2 dxdydz = \int\limits_0^h dz \iint\limits_{\substack{\frac{z}{\beta} \le x \le \frac{z}{\alpha} \\ \sqrt{\frac{z}{b}} \le y \le \sqrt{\frac{z}{a}}}} x^2 dxdy = \int\limits_0^h dz \int\limits_{\frac{z}{\beta}}^{\frac{z}{\alpha}} x^2 dx \int\limits_{\sqrt{\frac{z}{b}}}^{\sqrt{\frac{z}{a}}} dy =$$

$$= \frac{1}{3}\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right)\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right)\int\limits_0^h z^3 \sqrt{z}\, dz = \frac{2}{27} h^4 \sqrt{h}\left(\frac{1}{\alpha^3} - \frac{1}{\beta^3}\right)\left(\frac{1}{\sqrt{a}} - \frac{1}{\sqrt{b}}\right).$$

Найдите объемы тел, ограниченных следующими поверхностями:

4102. $z = x + y, z = xy, x + y = 1, x = 0, y = 0.$

$$V = \iint\limits_{\substack{x, y \ge 0 \\ x+y \le 1}} (x + y - xy)\,dxdy = \frac{1}{3} - \int\limits_0^1 x\,dx \int\limits_0^{1-x} y\,dy = \frac{1}{3} - \frac{1}{2}\int\limits_0^1 x(1-x)^2\,dx = \frac{1}{3} - \frac{1}{6}\cdot\frac{1}{4} = \frac{1}{3} - \frac{1}{24} = \frac{7}{24}.$$

4108. $\left(x^2 + y^2 + z^2\right)^2 = a^2\left(x^2 + y^2 - z^2\right).$

$$V = \iiint\limits_{V} dxdydz = \iiint\limits_{\substack{V_1:0\le\varphi\le 2\pi \\ -\frac{\pi}{4}\le\psi\le\frac{\pi}{4} \\ \rho^2 \le a^2 \cos 2\psi}} \rho^2 \cos\psi\, d\rho\, d\varphi\, d\psi =$$

$$2\pi\cdot 2 \int\limits_0^{\pi/4} \cos\psi\, d\psi \int\limits_0^{a\sqrt{\cos 2\psi}} \rho^2 d\rho = \frac{4\pi a^3}{3}\int\limits_0^{\pi/4} \cos\psi \cos^{3/2} 2\psi\, d\psi =$$

$$= \frac{4\pi a^3}{3}\int\limits_0^{1/\sqrt{2}}\left(1 - 2u^2\right)^{3/2} du = \frac{4\pi a^3}{3\sqrt{2}}\int\limits_0^1\left(1 - v^2\right)^{3/2} dv =$$

$$= \frac{4\pi a^3}{3\sqrt{2}}\int\limits_0^{\pi/2}\cos^4 t\, dt = \frac{4\pi a^3}{3\sqrt{2}}\frac{1\cdot 3}{2\cdot 4}\frac{\pi}{2} = \frac{\pi^2 a^3}{4\sqrt{2}}.$$

4118 б). $\sqrt[3]{\dfrac{x}{a}} + \sqrt[3]{\dfrac{y}{b}} + \sqrt[3]{\dfrac{z}{c}} = 1,\ \ a,b,c > 0;\ x,y,z \ge 0.$

$$V = \iiint\limits_{\substack{x,y,z\ge 0 \\ \sqrt[3]{\frac{x}{a}}+\sqrt[3]{\frac{y}{b}}+\sqrt[3]{\frac{z}{c}}\le 1}} dxdydz = \begin{bmatrix} x = au \\ y = bv \\ z = cw \end{bmatrix} = abc \iiint\limits_{\substack{u,v,w\ge 0 \\ \sqrt[3]{u}+\sqrt[3]{v}+\sqrt[3]{w}\le 1}} dxdydz = \begin{bmatrix} u = \xi^3 \\ v = \eta^3 \\ w = \zeta^3 \end{bmatrix} =$$

$$= 27abc \iiint\limits_{\substack{\xi,\eta,\zeta\ge 0 \\ \zeta+\eta+\zeta\le 1}} \xi^2\eta^2\zeta^2 d\xi d\eta d\zeta = 27abc\int\limits_0^1 \xi^2 d\xi \int\limits_0^{1-\xi}\eta^2 d\eta \int\limits_0^{1-\xi-\eta}\zeta^2 d\zeta =$$

$$= 9abc\int\limits_0^1 \xi^2 d\xi \int\limits_0^{1-\xi}\eta^2\left(1-\xi-\eta\right)^3 d\eta = 9abc\frac{3}{3}\frac{2}{4}\frac{1}{5}\frac{1}{6}\int\limits_0^1 \xi^2\left(1-\xi\right)^6 d\xi =$$

$$= \frac{3}{20}abc\int\limits_0^1 \xi^2\left(1-\xi\right)^6 d\xi = \frac{3}{20}abc\frac{2}{7}\frac{1}{8}\frac{1}{9} = \frac{1}{20}abc\frac{2}{7}\frac{1}{8}\frac{1}{3} = \frac{abc}{1680}.$$

Вычислите тройной интеграл

$$\iiint\limits_V z^2 dxdydz \text{, } V \text{ определяется неравенствами } \begin{matrix} x^2+y^2+z^2\le 3, \\ x^2+y^2\le 2z^2, \\ z\ge 0. \end{matrix}$$

$$\iiint\limits_V z^2 dxdydz = \int\limits_0^1 z^2 dz \iint\limits_{x^2+y^2\le 2z^2} dxdy + \int\limits_1^{\sqrt3} z^2 dz \iint\limits_{x^2+y^2\le\sqrt{3-z^2}} dxdy =$$

$$= \pi\int\limits_0^1 2z^4 dz + \pi\int\limits_1^{\sqrt3} z^2\left(3-z^2\right)dz = \frac{2}{5}\pi + \pi\left(z^3-\frac{1}{5}z^5\right)\Bigg|_1^{\sqrt3} =$$

$$= \frac{2}{5}\pi + \pi\left(3\sqrt3-1-\frac{9\sqrt3}{5}+\frac{1}{5}\right) = \frac{6\sqrt3-2}{5}\pi$$

или

$$\iiint\limits_V z^2 dxdydz = \iint\limits_{x^2+y^2\le 2} dxdy \int\limits_{\sqrt{\frac{x^2+y^2}{2}}}^{\sqrt{3-x^2-y^2}} z^2 dz =$$

$$= \frac{1}{3}\iint\limits_{x^2+y^2\le 2}\left(\left(3-x^2-y^2\right)^{3/2}-\frac{\left(x^2+y^2\right)^{3/2}}{2\sqrt2}\right)dxdy =$$

$$= \frac{1}{3}2\pi\int\limits_0^{\sqrt2}\left(\left(3-r^2\right)^{3/2}-\frac{r^3}{2\sqrt2}\right)rdr = \frac{2\pi}{3}\left(\left(-\frac{1}{5}\left(3-r^2\right)^{5/2}-\frac{r^5}{10\sqrt2}\right)\right)\Bigg|_0^{\sqrt2} =$$

$$= \frac{2\pi}{3}\left(\frac{9\sqrt3}{5}-\frac{1}{5}-\frac{2}{5}\right) = 2\pi\left(\frac{3\sqrt3}{5}-\frac{1}{5}\right)$$

4084.

$$\int\limits_0^x d\xi \int\limits_0^\xi d\eta \int\limits_0^\eta f(\zeta) d\zeta = \iiint\limits_{0 \le \zeta \le \eta \le \xi \le x} f(\zeta) d\xi d\eta d\zeta = \int\limits_0^x f(\zeta) d\zeta \iint\limits_{\zeta \le \eta \le \xi \le x} d\xi d\eta = \int\limits_0^x f(\zeta) d\zeta \int\limits_\zeta^x d\eta \int\limits_\eta^x d\xi =$$

$$= \int\limits_0^x f(\zeta) d\zeta \int\limits_\zeta^x (x-\eta) d\eta = \frac{1}{2} \int\limits_0^x f(\zeta)(x-\zeta)^2 d\zeta.$$

Вычислите тройные интегралы:

4076.

$$\iiint\limits_V xy^2 z^3 dx dy dz = \iint\limits_{0 \le y \le x \le 1} xy^2 dx dy \int\limits_0^{xy} z^3 dz = \frac{1}{4} \iint\limits_{0 \le y \le x \le 1} x^5 y^6 dx dy = \frac{1}{4} \int\limits_0^1 x^5 dx \int\limits_0^x y^6 dy = \frac{1}{28} \int\limits_0^1 x^{12} dx = \frac{1}{364},$$

тело $V$ ограничено поверхностями $z = xy, y = x, x = 1, z = 0$.

4087. $\iiint\limits_V \sqrt{x^2 + y^2 + z^2} dx dy dz$, тело $V$ ограничено поверхностью $x^2 + y^2 + z^2 = z$.

$$\iiint\limits_V \sqrt{x^2 + y^2 + z^2} dx dy dz = \left[ x = \rho\cos\varphi\cos\psi, y = \rho\sin\varphi\cos\psi, z = \rho\sin\psi \right] =$$

$$= \iiint\limits_{\substack{0 \le \varphi \le 2\pi \\ 0 \le \psi \le \frac{\pi}{2} \\ 0 \le \rho \le \cos\psi}} \rho^3 \cos\psi \, d\rho d\varphi d\psi = 2\pi \int\limits_0^{\pi/2} \cos\psi \, d\psi \int\limits_0^{\sin\psi} \rho^3 d\rho = \frac{\pi}{2} \int\limits_0^{\pi/2} \cos\psi \sin^4\psi \, d\psi = \frac{\pi}{10}.$$

Найдите объемы тел, ограниченных следующими поверхностями:

4118 a). $\sqrt{\dfrac{x}{a}} + \sqrt{\dfrac{y}{b}} + \sqrt{\dfrac{z}{c}} = 1 \ (x, y, z \ge 0)$.

$$V = \iiint\limits_V dx dy dz = adc \iiint\limits_{V_1: u,v,w \ge 0, \sqrt{u}+\sqrt{v}+\sqrt{w} \le 1} du dv dw = \left[ u = \xi^2, v = \eta^2, w = \zeta^2 \right] =$$

$$= 8 \iiint\limits_{V_1: \xi,\eta,\zeta \ge 0, \xi+\eta+\zeta \le 1} \xi\eta\zeta \, d\xi d\eta d\zeta = 8 \int\limits_0^1 \xi d\xi \int\limits_0^{1-\xi} \eta d\eta \int\limits_0^{1-\xi-\eta} \zeta d\zeta = 4 \int\limits_0^1 \xi d\xi \int\limits_0^{1-\xi} \eta(1-\xi-\eta)^2 d\eta =$$

$$= 4 \int\limits_0^1 \xi d\xi \int\limits_0^{1-\xi} \eta(1-\xi-\eta)^2 d\eta = \frac{4}{3} \int\limits_0^1 \xi d\xi \int\limits_0^{1-\xi} (1-\xi-\eta)^3 d\eta = \frac{1}{3} \int\limits_0^1 \xi(1-\xi)^4 d\xi = \frac{1}{15} \int\limits_0^1 (1-\xi)^5 d\xi = \frac{1}{90}.$$