Геометрические приложения определенного интеграла

Вычисление площадей

2397

$$ax = y^2,\ ay = x^2;$$

$$S = \int_0^a \left(\sqrt{ax} - \frac{x^2}{a}\right)dx = \left(\frac{2}{3} - \frac{1}{3}\right)a^2 = \frac{1}{3}a^2$$

2406

$$Ax^2 + 2Bxy + Cy^2 = 1,\ A > 0,\ AC - B^2 > 0;$$

$$y = \frac{-Bx \pm \sqrt{\left(B^2 - AC\right)x^2 + C}}{C}$$

$$S = \frac{4}{C}\int_0^{\sqrt{\frac{C}{AC-B^2}}} \sqrt{\left(B^2 - AC\right)x^2 + C}\,dx = \left[x = \sqrt{\frac{C}{AC - B^2}}u\right] =$$

$$= \frac{4}{\sqrt{AC - B^2}}\int_0^1 \sqrt{1 - u^2}\,du = \frac{\pi}{\sqrt{AC - B^2}}$$

2407

$$y^2 = \frac{x^3}{2a - x},\ x = 2a;$$

$$y = \pm\sqrt{\frac{x^3}{2a - x}};$$

$$S = 2\int_0^{2a} \frac{x^{3/2}}{\left(2a - x\right)^{1/2}}dx = \left[x = a\left(1 - \cos t\right)\right] = 2a^2\int_0^{\pi} \frac{\left(1 - \cos t\right)^{3/2}}{\left(1 + \cos t\right)^{1/2}}\sin t\,dt =$$

$$= 2a^2\int_0^{\pi} 4\sin^4\frac{t}{2}dt = 16a^2\int_0^{\pi/2} \sin^4 u\,du = 16a^2\frac{3!!}{4!!}\frac{\pi}{2} = 3\pi a^2$$

2414

$$x = 2t - t^2,\ y = 2t^2 - t^3;$$

$$S = \frac{1}{2}\int_0^2 \left(\left(2t - t^2\right)\left(4t - 3t^2\right) - \left(2 - 2t\right)\left(2t^2 - t^3\right)\right) dt =$$

$$\frac{1}{2}\int_0^2 t^2\left(2 - t\right)\left(\left(4 - 3t\right) - \left(2 - 2t\right)\right) dt = \frac{1}{2}\int_0^2 t^2\left(2 - t\right)^2 dt =$$

$$= \frac{1}{6}t^3\left(2 - t\right)^2\Big|_0^2 + \frac{1}{3}\int_0^2 t^3\left(2 - t\right) dt = \frac{1}{12}\int_0^2 t^4 dt = \frac{1}{60}2^5 = \frac{8}{15}$$

2419

$$r = a\left(1 + \cos\varphi\right);$$

$$S = \int_0^\pi a^2\left(1 + \cos\varphi\right)^2 d\varphi = \int_0^\pi a^2\left(1 + 2\cos\varphi + \cos^2\varphi\right) d\varphi =$$

$$= a^2\left(\pi + \frac{\pi}{2}\right) = \frac{3}{2}\pi a^2$$

2422 a)

$$r = \frac{p}{1 + \varepsilon\cos\varphi};$$

$$S = \int_0^\pi \frac{p^2}{\left(1 + \varepsilon\cos\varphi\right)^2} d\varphi = \left[z = \operatorname{tg}\frac{\varphi}{2}\right] = p^2\int_0^{\pi/2} \frac{1}{\left(1 + \varepsilon\frac{1 - z^2}{1 + z^2}\right)^2}\frac{2dz}{1 + z^2} =$$

$$= 2p^2\int_0^{+\infty} \frac{1 + z^2}{\left(\left(1 + \varepsilon\right) + \left(1 - \varepsilon\right)z^2\right)^2} dz = \left[z = \sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}w\right] =$$

$$= 2p^2\frac{1}{\left(1 + \varepsilon\right)^2}\sqrt{\frac{1 + \varepsilon}{1 - \varepsilon}}\int_0^{+\infty} \frac{1 + \frac{1 + \varepsilon}{1 - \varepsilon}w^2}{\left(1 + w^2\right)^2} dw = 2p^2\frac{2}{\left(1 - \varepsilon^2\right)^{3/2}}\frac{\pi}{4} = \frac{\pi p^2}{\left(1 - \varepsilon^2\right)^{3/2}}$$

2427

$$x^4+y^4=a^2\left(x^2+y^2\right);$$

$$r^4\left(\cos^4\varphi+\sin^4\varphi\right)=a^2r^2,\ r^2=\frac{a^2}{\cos^4\varphi+\sin^4\varphi};$$

$$S=2\int_0^{\pi/2}\frac{a^2}{\cos^4\varphi+\sin^4\varphi}d\varphi=4\int_0^{\pi/4}\frac{a^2}{1-\frac{1}{2}\sin^2 2\varphi}d\varphi=$$

$$=2\int_0^{\pi/2}\frac{a^2}{1-\frac{1}{2}\sin^2\psi}d\psi=4\int_0^{\pi/2}\frac{a^2}{2-\sin^2\psi}d\psi=4\int_0^{\pi/2}\frac{a^2}{\sin^2\psi+2\cos^2\psi}d\psi=$$

$$=4\int_0^{\pi/2}\frac{a^2}{\mathrm{tg}^2\,\psi+2}\frac{d\psi}{\cos^2\psi}=4a^2\int_0^{+\infty}\frac{dz}{z^2+2}=4a^2\frac{1}{\sqrt{2}}\frac{\pi}{2}=\sqrt{2}\pi a^2$$

Вычисление длин дуг

2431

$$y=x^{3/2},\ 0\le x\le 4;$$

$$L=\int_0^4\sqrt{1+\left(\frac{3}{2}x^{1/2}\right)^2}dx=\int_0^4\sqrt{1+\frac{9}{4}x}dx=\frac{4}{9}\int_0^9\sqrt{1+u}du=$$

$$=\frac{4}{9}\frac{2}{3}(1+u)^{3/2}\Big|_0^9=\frac{8}{27}\left(10\sqrt{10}-1\right)$$

2437

$$y=\ln\cos x,\ 0\le x\le a<\frac{\pi}{2}$$

$$L=\int_0^a\sqrt{1+\mathrm{tg}^2\,x}dx=\int_0^a\frac{dx}{\cos x}=\ln\mathrm{tg}\left(\frac{\pi}{4}+\frac{x}{2}\right)\Big|_0^a=\ln\mathrm{tg}\left(\frac{\pi}{4}+\frac{a}{2}\right)$$

2443

$$x=a\left(t-\sin t\right),\ y=a\left(1-\cos t\right)$$

$$L=a\int_0^{2\pi}\sqrt{\left(1-\cos t\right)^2+\sin^2 t}dt=a\int_0^{2\pi}\sqrt{2-2\cos t}dt=$$

$$=2a\int_0^{2\pi}\sin\frac{t}{2}dt=-4a\cos\frac{t}{2}\Big|_0^{2\pi}=8a$$

Вычисление объемов

2456

$$a(z)=c+(a-c)\frac{z}{h},\ b(z)=b\frac{z}{h},$$

$$S(z)=bc\frac{z}{h}+(a-c)b\frac{z^2}{h^2};$$

$$V=\int_0^h S(z)dz=\frac{b\,\mathrm{ch}}{2}+\frac{(a-c)bh}{3}=\frac{(2a+c)bh}{6}$$

2462

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ \ z=\frac{c}{a}x$$

$$S(z)=2\int_{\frac{a}{c}z}^{a} b\sqrt{1-\frac{x^2}{a^2}}dx=2ab\int_{\frac{z}{c}}^{1}\sqrt{1-u^2}du=$$

$$=ab\left(u\sqrt{1-u^2}+\arcsin u\right)\Big|_{\frac{z}{c}}^{1}=ab\left(-\frac{z}{c}\sqrt{1-\frac{z^2}{c^2}}+\arccos\frac{z}{c}\right);$$

$$V=ab\int_0^c\left(-\frac{z}{c}\sqrt{1-\frac{z^2}{c^2}}+\arccos\frac{z}{c}\right)dz=abc\int_0^1\left(-u\sqrt{1-u^2}+\arccos u\right)du=$$

$$=abc\left(u\arccos u\Big|_0^1+\int_0^1\left(\frac{u}{\sqrt{1-u^2}}-u\sqrt{1-u^2}\right)du\right)=abc\int_0^1\frac{u^3}{\sqrt{1-u^2}}du=$$

$$=abc\int_0^{\pi/2}\sin^3 t dt=\frac{2}{3}abc$$

$$S(x)=2b\sqrt{1-\frac{x^2}{a^2}}\frac{c}{a}x;$$

$$V=2bc\int_0^a\sqrt{1-\frac{x^2}{a^2}}\frac{x}{a}dx=2abc\int_0^1\sqrt{1-u^2}udu=\frac{2}{3}abc$$

2473

$$y=2x-x^2,\ y=0$$

$$a)\ V=\pi\int_0^2\left(2x-x^2\right)^2dx=\pi\int_0^2x^2(2-x)^2dx=\pi\frac{2}{3}\frac{1}{4}\frac{2^5}{5}=\frac{16\pi}{15}$$

$$b)\ V=2\pi\int_0^2x\left(2x-x^2\right)dx=2\pi\int_0^2x^2(2-x)dx=2\pi\frac{1}{3}\frac{2^4}{4}=\frac{8\pi}{3}$$

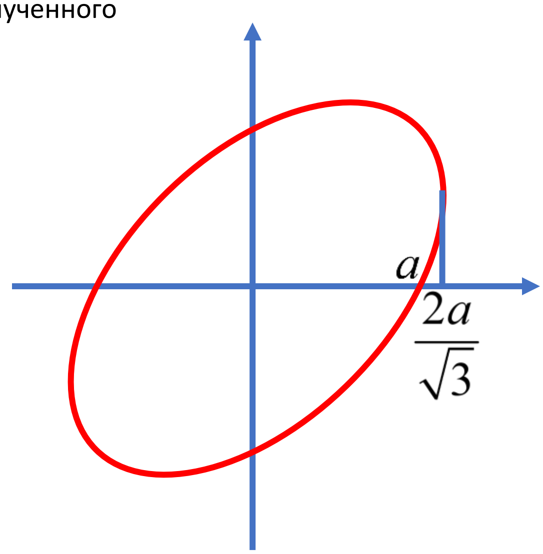
2478 Найти объем тела, полученного

Вращением фигуры

$x, y \ge 0$

$x^2 - xy + y^2 \le a^2$

вокруг оси абсцисс.



$x^2 - xy + y^2 = a^2,\ x \ge 0;$

$$\begin{cases} x = \frac{1}{\sqrt{2}}(u - v) \\ y = \frac{1}{\sqrt{2}}(u + v) \end{cases}$$

$$\frac{1}{2}\left(u^2 + 3v^2\right) = a^2$$

$$\begin{cases} u = \sqrt{2}a\cos t \\ v = \sqrt{\frac{2}{3}}a\sin t \end{cases} \quad \begin{cases} x = a\left(\cos t - \frac{1}{\sqrt{3}}\sin t\right) \\ y = a\left(\cos t + \frac{1}{\sqrt{3}}\sin t\right) \end{cases}$$

$$V = \pi a^3 \int_{-\pi/3}^{\pi/3} \left(\cos^2 t + \frac{2}{\sqrt{3}}\cos t \sin t + \frac{1}{3}\sin^2 t\right)\left(\sin t + \frac{1}{\sqrt{3}}\cos t\right)dt =$$

$$= \pi a^3 \int_{-\pi/3}^{\pi/3} \left(\frac{1}{\sqrt{3}}\cos^3 t + \frac{2}{\sqrt{3}}\cos t \sin^2 t + \frac{1}{3\sqrt{3}}\sin^2 t \cos t\right)dt =$$

$$= 2\pi a^3 \int_{0}^{\pi/3} \left(\frac{1}{\sqrt{3}}\cos^3 t + \frac{7}{3\sqrt{3}}\cos t \sin^2 t\right)dt =$$

$$= 2\pi a^3 \int_{0}^{\sqrt{3}/2} \left(\frac{1}{\sqrt{3}}\left(1 - u^2\right) + \frac{7}{3\sqrt{3}}u^2\right)du = 2\pi a^3 \frac{1}{3\sqrt{3}} \int_{0}^{\sqrt{3}/2} \left(3 + 4u^2\right)du =$$

$$= 2\pi a^3 \frac{1}{3\sqrt{3}}\left(3u + \frac{4}{3}u^3\right)\Bigg|_0^{\sqrt{3}/2} = 2\pi a^3 \frac{1}{3\sqrt{3}}\left(\frac{3\sqrt{3}}{2} + \frac{\sqrt{3}}{2}\right) = 2\pi a^3 \frac{2}{3} = \frac{4\pi a^3}{3}$$

или $y=\dfrac{x\pm\sqrt{x^2-4\left(x^2-a^2\right)}}{2}=\dfrac{x\pm\sqrt{4a^2-3x^2}}{2}$

$$V=\pi\left(\int_0^{2a/\sqrt{3}}\left(\frac{x+\sqrt{4a^2-3x^2}}{2}\right)^2-\int_a^{2a/\sqrt{3}}\left(\frac{x-\sqrt{4a^2-3x^2}}{2}\right)^2\right)$$

$$\int_0^a\left(\frac{x+\sqrt{4a^2-3x^2}}{2}\right)^2dx=\frac{1}{4}\int_0^a\left(x^2+2x\sqrt{4a^2-3x^2}+4a^2-3x^2\right)dx=$$

$$\frac{1}{4}\int_0^a\left(4a^2-2x^2+2x\sqrt{4a^2-3x^2}\right)dx=\frac{1}{4}\left(4a^2x-\frac{2}{3}x^3-\frac{2}{9}\left(4a^2-3x^2\right)^{3/2}\right)\Bigg|_0^a=$$

$$=\frac{1}{4}a^3\left(4-\frac{2}{3}-\frac{2}{9}+\frac{16}{9}\right)=\frac{11}{9}a^3$$

$$\int_a^{2a/\sqrt{3}}x\sqrt{4a^2-3x^2}dx=-\frac{1}{9}\left(4a^2-3x^2\right)^{3/2}\Bigg|_a^{2a/\sqrt{3}}=\frac{1}{9}a^3$$

$$V=\frac{11}{9}a^3+\frac{1}{9}a^3=\frac{4}{3}a^3.$$

2480

$$x=a\left(t-\sin t\right)$$
$$y=a\left(1-\cos t\right)$$

$$V_1=\pi a^3\int_0^{2\pi}\left(1-\cos t\right)^3dt=\pi a^3\int_0^{2\pi}\left(1-3\cos t+3\cos^2 t-\cos^3 t\right)dt=$$

$$=\pi\left(2\pi+0+3\pi+0\right)=5\pi^2a^3;$$

$$V_2=2\pi a^3\int_0^{2\pi}\left(t-\sin t\right)\left(1-\cos t\right)^2dt=2\pi a^3\int_0^{2\pi}t\left(1-\cos t\right)^2dt=$$

$$=2\pi a^3\int_0^{2\pi}t\left(\frac{3}{2}-2\cos t+\frac{1}{2}\cos 2t\right)dt=3\pi a^3\int_0^{2\pi}tdt=3\pi a^3 2\pi^2=6\pi^3a^3$$

$$V_3=8\pi^2a^3-\pi a^3\int_0^{2\pi}\left(1+\cos t\right)^2\left(1-\cos t\right)dt=8\pi^2a^3-\pi a^3\int_0^{2\pi}\left(1-\cos^2 t\right)\left(1+\cos t\right)dt=$$

$$=8\pi^2a^3-\pi a^3\int_0^{2\pi}\left(1-\cos^2 t+\cos t-\cos^3 t\right)dt=8\pi^2a^3-\pi^2a^3=7\pi^2a^3.$$

2483.2 а)

$$\left(x^2+y^2\right)^2=a^2\left(x^2-y^2\right)$$

$$r^2=a^2\cos 2\varphi$$

$$V=\frac{4\pi}{3}a^3\int\limits_0^{\pi/4}\cos^{3/2}2\varphi\sin\varphi d\varphi=\frac{4\pi}{3}a^3\int\limits_{1/\sqrt{2}}^{1}\left(2u^2-1\right)^{3/2}du=\left[v=\sqrt{2}u\right]=\frac{2\sqrt{2}\pi}{3}a^3\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{3/2}du$$

$$=\left[u=\operatorname{ch}t\right]=\frac{2\sqrt{2}\pi}{3}a^3\int\limits_0^{t_0}\operatorname{sh}^4 t dt=\frac{2\sqrt{2}\pi}{3}a^3\frac{1}{4}\int\limits_0^{t_0}\left(\operatorname{ch}2t-1\right)^2dt=\frac{\pi}{3\sqrt{2}}a^3\int\limits_0^{t_0}\left(\operatorname{ch}^2 2t-2\operatorname{ch}2t+1\right)dt=$$

$$=\frac{\pi}{3\sqrt{2}}a^3\int\limits_0^{t_0}\left(\frac{1}{2}\operatorname{ch}4t-2\operatorname{ch}2t+\frac{3}{2}\right)dt=\frac{\pi}{3\sqrt{2}}a^3\left(\frac{1}{8}\operatorname{sh}4t_0-\operatorname{sh}2t_0+\frac{3}{2}t_0\right)$$

$$\operatorname{ch}t=\sqrt{2},e^t+e^{-t}=2\sqrt{2},\ e^{2t}-2\sqrt{2}e^t+1=0,\ e^t=\sqrt{2}+1,\ t=\ln\left(\sqrt{2}+1\right)$$

$$\operatorname{sh}t_0=1,\ \operatorname{sh}2t_0=2\sqrt{2},\ \operatorname{ch}2t_0=3,\ \operatorname{sh}4t_0=12\sqrt{2};$$

$$V=\frac{\pi}{3\sqrt{2}}a^3\left(\frac{3}{2}\sqrt{2}-2\sqrt{2}+\frac{3}{2}\ln\left(\sqrt{2}+1\right)\right)=\frac{\pi}{3\sqrt{2}}a^3\left(-\frac{1}{\sqrt{2}}+\frac{3}{2}\ln\left(\sqrt{2}+1\right)\right)=\pi a^3\left(-\frac{1}{6}+\frac{1}{2\sqrt{2}}\ln\left(\sqrt{2}+1\right)\right)$$

или

$$I=\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{3/2}du=\int\limits_1^{\sqrt{2}}u^2\left(u^2-1\right)^{1/2}du-\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du=\frac{u^3}{3}\left(u^2-1\right)^{1/2}\Bigg|_0^{\sqrt{2}}-\frac{1}{3}\int\limits_1^{\sqrt{2}}\frac{u^4}{\left(u^2-1\right)^{1/2}}du-\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du=$$

$$=\frac{2\sqrt{2}}{3}-\frac{1}{3}\int\limits_1^{\sqrt{2}}\frac{\left(u^2-1\right)^2+2\left(u^2-1\right)+1}{\left(u^2-1\right)^{1/2}}du-\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du=\frac{2\sqrt{2}}{3}-\frac{1}{3}I-\frac{5}{3}\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du-\frac{1}{3}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du;$$

$$I=\frac{1}{\sqrt{2}}-\frac{5}{4}\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du-\frac{1}{4}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du;$$

$$\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du=u\left(u^2-1\right)^{1/2}\Bigg|_1^{\sqrt{2}}-\int\limits_1^{\sqrt{2}}\frac{u^2}{\left(u^2-1\right)^{1/2}}du=\sqrt{2}-\int\limits_1^{\sqrt{2}}\frac{u^2-1+1}{\left(u^2-1\right)^{1/2}}du=\sqrt{2}-\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du-\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du;$$

$$\int\limits_1^{\sqrt{2}}\left(u^2-1\right)^{1/2}du=\frac{1}{\sqrt{2}}-\frac{1}{2}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du;$$

$$I=\frac{1}{\sqrt{2}}-\frac{5}{4}\left(\frac{1}{\sqrt{2}}-\frac{1}{2}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du\right)-\frac{1}{4}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du=$$

$$=-\frac{1}{4\sqrt{2}}+\frac{3}{8}\int\limits_1^{\sqrt{2}}\frac{1}{\left(u^2-1\right)^{1/2}}du=-\frac{1}{4\sqrt{2}}+\frac{3}{8}\ln\left(u+\sqrt{u^2-1}\right)\Bigg|_1^{\sqrt{2}}=-\frac{1}{4\sqrt{2}}+\frac{3}{8}\ln\left(\sqrt{2}+1\right)$$

Вычисление площадей поверхностей вращения

2490

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,\ y=b\sqrt{1-\frac{x^2}{a^2}}$$

$$S=4\pi\int_0^a b\sqrt{1-\frac{x^2}{a^2}}\sqrt{1+b^2\frac{x^2}{\left(1-\frac{x^2}{a^2}\right)a^4}}dx=4\pi b\int_0^a\sqrt{1-\frac{x^2}{a^2}+b^2\frac{x^2}{a^4}}dx=4\pi ab\int_0^1\sqrt{1-u^2+\frac{b^2}{a^2}u^2}du=$$

$$=4\pi ab\int_0^1\sqrt{1-\left(1-\frac{b^2}{a^2}\right)u^2}du;$$

$a>b$

$$S=4\pi ab\int_0^1\sqrt{1-\varepsilon^2u^2}du=\frac{4\pi ab}{\varepsilon}\int_0^\varepsilon\sqrt{1-v^2}dv=\frac{2\pi ab}{\varepsilon}\left(v\sqrt{1-v^2}+\arcsin v\right)\Big|_0^\varepsilon=$$

$$=\frac{2\pi ab}{\varepsilon}\left(\varepsilon\sqrt{1-\varepsilon^2}+\arcsin\varepsilon\right)=2\pi ab\left(\sqrt{1-\varepsilon^2}+\frac{\arcsin\varepsilon}{\varepsilon}\right)=2\pi b^2+2\pi ab\frac{\arcsin\varepsilon}{\varepsilon};$$

$a<b$

$$S=4\pi ab\int_0^1\sqrt{1+\left(\frac{b^2}{a^2}-1\right)u^2}du=\left[\alpha=\frac{\sqrt{b^2-a^2}}{a}\right]=\frac{4\pi ab}{\alpha}\int_0^\alpha\sqrt{1+v^2}dv=\frac{2\pi ab}{\alpha}\left(\alpha\sqrt{1+\alpha^2}+\ln\left(\alpha+\sqrt{1+\alpha^2}\right)\right)=$$

$$=2\pi ab\sqrt{1+\alpha^2}+2\pi ab\frac{\ln\left(\alpha+\sqrt{1+\alpha^2}\right)}{\alpha}=2\pi b^2+\frac{2\pi a^2}{\varepsilon}\ln\left(\frac{b}{a}(\varepsilon+1)\right);$$

$$\alpha=\varepsilon\frac{b}{a}$$

2492

$$x^{2/3}+x^{2/3}=a^{2/3};$$

$$x=a\cos^3 t$$

$$y=a\sin^3 t$$

$$S=4\pi\int_0^{\pi/2}a\sin^3 t\sqrt{\left(-3a\cos^2 t\sin t\right)^2+\left(-3a\sin^2 t\cos t\right)^2}dt=12\pi a^2\int_0^{\pi/2}\sin^4 t\cos t dt=\frac{12}{5}\pi a^2$$