Занятие 6 Определенный интеграл

2207 $\int\limits_0^{\pi} \sin x dx = 2$

2214

$$I = \int\limits_{-1}^{1} \frac{dx}{\sqrt{\left(1-2ax+a^2\right)\left(1-2bx+b^2\right)}},\ 0 < a,b < 1$$

$$I = -\frac{1}{\sqrt{ab}} \ln\left(\sqrt{b}\sqrt{1-2ax+a^2} + \sqrt{a}\sqrt{1-2bx+b^2}\right)\Bigg|_{-1}^{1} =$$

$$= \frac{1}{\sqrt{ab}} \ln \frac{\sqrt{b}\sqrt{1+2a+a^2} + \sqrt{a}\sqrt{1+2b+b^2}}{\sqrt{b}\sqrt{1-2a+a^2} + \sqrt{a}\sqrt{1-2b+b^2}} = \frac{1}{\sqrt{ab}} \ln \frac{\sqrt{b}\left(1+a\right)+\sqrt{a}\left(1+b\right)}{\sqrt{b}\left(1-a\right)+\sqrt{a}\left(1-b\right)} =$$

$$= \frac{1}{\sqrt{ab}} \ln \frac{\left(\sqrt{a}+\sqrt{b}\right)+\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)}{\left(\sqrt{a}+\sqrt{b}\right)-\sqrt{ab}\left(\sqrt{a}+\sqrt{b}\right)} = \frac{1}{\sqrt{ab}} \ln \frac{1+\sqrt{ab}}{1-\sqrt{ab}}$$

2221

$$\left(\frac{n}{n^2+1^2} + \frac{n}{n^2+2^2} + \cdots + \frac{n}{n^2+n^2}\right) = \lim_{n\to\infty} \frac{1}{n}\left(\frac{1}{1+\left(\frac{1}{n}\right)^2} + \frac{1}{1+\left(\frac{1}{n}\right)^2} + \cdots + \frac{1}{1+\left(\frac{n}{n}\right)^2}\right) =$$

$$= \int\limits_0^1 \frac{dx}{1+x^2} = \operatorname{arctg} x\Big|_0^1 = \frac{\pi}{4}$$

2241 $\int\limits_0^{2\pi} x^2 \cos x dx = x^2 \sin x\Big|_0^{2\pi} - 2\int\limits_0^{2\pi} x \sin x dx = 2x\cos x\Big|_0^{2\pi} - 2\int\limits_0^{2\pi} \cos x dx = 4\pi$

2243

$$\int\limits_0^1 \arccos x dx = x \arccos x\Big|_0^1 + \int\limits_0^1 \frac{dx}{\sqrt{1-x^2}} = -\sqrt{1-x^2}\Big|_0^1 = 1$$

2246

$$\int\limits_0^a x^2\sqrt{a^2-x^2}\,dx = \left[x = a\sin t\right] = \int\limits_0^{\pi/2} a^4 \sin^2 t \cos^2 t dt = \frac{a^4}{4}\int\limits_0^{\pi/2} \sin^2 2t dt = \frac{\pi a^4}{16}$$

2248

$$\int_0^{\ln 2}\sqrt{e^x-1}dx=\left[y=\sqrt{e^x-1},dy=\frac{e^xdx}{2\sqrt{e^x-1}}\right]=2\int_0^1\frac{y^2}{y^2+1}dy=2-\frac{\pi}{2}$$

2257

Доказать, что если f непрерывна на $[0,1]$, то

$$\int_0^{\pi/2} f(\cos x)dx=\int_0^{\pi/2} f(\sin x)dx$$

$$\int_0^{\pi} x\,f(\sin x)dx=\frac{\pi}{2}\int_0^{\pi} f(\sin x)dx$$

2274

$$\int_0^3\arcsin\sqrt{\frac{x}{1+x}}dx=x\arcsin\sqrt{\frac{x}{1+x}}\Bigg|_0^3-\int_0^3 x\frac{1}{\sqrt{1-\frac{x}{1+x}}2\sqrt{\frac{x}{1+x}}}\frac{dx}{(1+x)^2}=$$

$$=\pi-\int_0^3\frac{\sqrt{x}dx}{2(1+x)}=\left[y=\sqrt{x},x=y^2\right]=\pi-\int_0^{\sqrt{3}}\frac{y^2}{y^2+1}dy=\pi-\int_0^{\sqrt{3}}\left(1-\frac{1}{y^2+1}\right)dy=$$

$$=\pi-\sqrt{3}+\pi/3=\frac{4\pi}{3}-\sqrt{3}$$

2275

$$\int_0^{2\pi}\frac{dx}{(2+\cos x)(3+\cos x)}=\int_0^{2\pi}\left(\frac{1}{2+\cos x}-\frac{1}{3+\cos x}\right)dx=2\pi\left(\frac{1}{\sqrt{3}}-\frac{1}{2\sqrt{2}}\right)$$

$$\int_0^{2\pi}\frac{dx}{a+\cos x}=2\int_0^{\pi}\frac{dx}{a+\cos x}=\left[z=\operatorname{tg}x/2\right]=4\int_0^{+\infty}\frac{1}{a+\frac{1-z^2}{1+z^2}}\frac{dz}{1+z^2}=$$

$$=4\int_0^{+\infty}\frac{dz}{a+1+(a-1)z^2}=4\frac{1}{\sqrt{a^2-1}}\frac{\pi}{2}=4\frac{1}{\sqrt{a^2-1}}\frac{\pi}{2}=\frac{2\pi}{\sqrt{a^2-1}}$$

2286

$$I_n=\int_0^1 x^m(\ln x)^n\,dx=\frac{x^{m+1}}{m+1}(\ln x)^n\Big|_0^1-\int_0^1\frac{x^m}{m+1}n(\ln x)^{n-1}\,dx=-\frac{n}{m+1}I_{n-1}=$$

$$=(-1)^n\frac{n!}{(m+1)^n}I_0=(-1)^n\frac{n!}{(m+1)^{n+1}}$$

2290

$$I_{nm}=\int_0^{\pi/2}\sin^n x\cos^m x dx$$

$m\geq 2$

$$I_{nm}=\int_0^{\pi/2}\sin^n x\cos^m x dx=\frac{1}{n+1}\sin^{n+1}x\cos^{m-1}x\Big|_0^{\pi/2}+\frac{m-1}{n+1}I_{n+2,m-2}$$

m нечетно

$$I_{nm}=\frac{m-1}{n+1}\frac{m-3}{n+3}\cdots\frac{2}{n+m-2}I_{n+m-1,1}=\frac{m-1}{n+1}\frac{m-3}{n+3}\cdots\frac{2}{n+m-2}\frac{1}{n+m}=$$
$$=\frac{(m-1)!!(n-1)!!}{(n+m)!!}$$

n, m четны

$$I_{nm}=\frac{(m-1)!!(n-1)!!}{(n+m-1)!!}I_{n+m,0}=\frac{(m-1)!!(n-1)!!}{(n+m-1)!!}\frac{(n+m-1)!!}{(n+m)!!}\frac{\pi}{2}=\frac{(m-1)!!(n-1)!!}{(n+m)!!}\frac{\pi}{2}$$

2291

$$I_n=\int_0^{\pi}\frac{\sin nx}{\sin x}dx$$

$$I_{n+1}-I_{n-1}=\int_0^{\pi}\frac{\sin(n+1)x-\sin(n-1)x}{\sin x}dx=$$

$$=2\int_0^{\pi}\frac{\sin x\cos nx}{\sin x}dx=2\int_0^{\pi}\cos nx dx=0$$

$I_1=\pi,\ I_2=0$

$I_{2k-1}=\pi,\ I_{2k}=0$

2295

$$\int_0^{\pi} \sin^{n-1} x \cos(n+1) x dx = \int_0^{\pi} \left(\frac{e^{xi} - e^{-xi}}{2i} \right)^{n-1} \cos(n+1) x dx = \frac{1}{2(2i)^{n-1}} \int_0^{\pi} \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k e^{(n-1-2k)xi} \left(e^{(n+1)xi} + e^{-(n+1)xi} \right) dx =$$

$$= \frac{1}{2(2i)^{n-1}} \int_0^{\pi} \sum_{k=0}^{n-1} (-1)^k C_{n-1}^k \left(e^{(2n-2k)xi} + e^{(-2-2k)xi} \right) dx = 0,$$

поскольку $\int_0^{\pi} e^{2jxi} dx = 0$ при $j \neq 0$