Интегрирование иррациональных выражений

1929

$$\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}dx = \left[y=\sqrt[6]{x+1},\ x=y^6-1 \right] = \int \frac{1-y^3}{1+y^2}6y^5dy =$$

$$= -6\int \frac{y^8-y^5}{y^2+1}dy = -6\left(y^6-y^4-y^3+y^2+y-1-\frac{y-1}{y^2+1} \right)dy =$$

$$= -6\left(\frac{y^7}{7}-\frac{y^5}{5}-\frac{y^4}{4}+\frac{y^3}{3}+\frac{y^2}{2}-y-\frac{1}{2}\ln\left(y^2+1\right)+\operatorname{arctg} y \right)+C$$

1932

$$\int \frac{dx}{\sqrt[3]{(x+1)^2(x-1)^4}} = \int \frac{dx}{(x-1)^2\sqrt[3]{\frac{(x+1)^2}{(x-1)^2}}} = \left[y=\sqrt[3]{\frac{x+1}{x-1}},\ dy=\frac{1}{3\left(\frac{x+1}{x-1}\right)^{2/3}}\frac{-2dx}{(x-1)^2} \right] =$$

$$= -\frac{3}{2}\int dy = -\frac{3}{2}y+C = -\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}+C$$

1940

$$\int \frac{\sqrt{x^2+2x+2}}{x}dx = \int \frac{x^2+2x+2}{x\sqrt{x^2+2x+2}}dx = \int \left(\frac{x+2}{\sqrt{x^2+2x+2}}+\frac{2}{x\sqrt{x^2+2x+2}} \right)dx;$$

$$\int \frac{x+2}{\sqrt{x^2+2x+2}}dx = \sqrt{x^2+2x+2}+\ln\left(x+1+\sqrt{x^2+2x+2}\right)+C,$$

$$\int \frac{dx}{x\sqrt{x^2+2x+2}} = \left[x=\frac{1}{y} \right] = -\int \frac{dy}{\sqrt{2y^2+2y+1}} = -\frac{1}{\sqrt{2}}\int \frac{dy}{\sqrt{y^2+y+\frac{1}{2}}} =$$

$$= -\frac{1}{\sqrt{2}}\ln\left(y+\frac{1}{2}+\sqrt{y^2+y+\frac{1}{2}} \right)+C_1 = -\frac{1}{\sqrt{2}}\ln\left(2y+1+\sqrt{2}\sqrt{2y^2+2y+1}\right)+C =$$

$$= -\frac{1}{\sqrt{2}}\ln\frac{x+2+\sqrt{2}\sqrt{x^2+2x+2}}{x}+C$$

1943

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx = \left(Ax^2+Bx+C\right)\sqrt{1+2x-x^2} + \lambda \int \frac{dx}{\sqrt{1+2x-x^2}}$$

$$\frac{x^3}{\sqrt{1+2x-x^2}} = \sqrt{1+2x-x^2} + \left(Ax^2+Bx+C\right)\frac{-x+1}{\sqrt{1+2x-x^2}} + \lambda \frac{1}{\sqrt{1+2x-x^2}}$$

$$x^3 = \left(2Ax+B\right)\left(-x^2+2x+1\right) + \left(Ax^2+Bx+C\right)\left(-x+1\right) + \lambda$$

$$\begin{cases} -3A = 1, \ \ A = -\dfrac{1}{3} \\ 5A - 2B = 0, \ \ \ B = -\dfrac{5}{6} \\ 2A + 3B - C = 0, \ \ \ C = -\dfrac{2}{3} - \dfrac{5}{2} = -\dfrac{19}{6} \\ B + C + \lambda = 0, \ \ \lambda = 4 \end{cases}$$

$$\int \frac{x^3}{\sqrt{1+2x-x^2}}dx = \left(-\frac{1}{3}x^2 - \frac{5}{6}x - \frac{19}{6}\right)\sqrt{1+2x-x^2} + 4\int \frac{dx}{\sqrt{2-\left(x-1\right)^2}} =$$

$$= -\frac{1}{6}\left(2x^2+5x+19\right)\sqrt{1+2x-x^2} + 4\arcsin\frac{x-1}{\sqrt{2}} + C$$

1957

$$\int \frac{dx}{\left(1+x^2\right)\sqrt{1-x^2}} = \left[\begin{array}{l} y = -\dfrac{x}{\sqrt{1-x^2}} = \left(\sqrt{1-x^2}\right)', \ y\sqrt{1-x^2} = -x, \\ \sqrt{1-x^2}dy + y^2 dx = -dx, \ \dfrac{dy}{y^2+1} = -\dfrac{dx}{\sqrt{1-x^2}}; \\ y^2 = \dfrac{x^2}{1-x^2}, \ 1+y^2 = \dfrac{1}{1-x^2} \end{array}\right] = -\int \frac{dy}{\left(2-\dfrac{1}{1+y^2}\right)\left(1+y^2\right)} =$$

$$= -\int \frac{1}{2y^2+1}dy = -\frac{1}{\sqrt{2}}\operatorname{arctg}\sqrt{2}y + C = \frac{1}{\sqrt{2}}\operatorname{arctg}\frac{\sqrt{2}x}{\sqrt{1-x^2}} + C$$

1963

$$\int \frac{x+1}{\left(x^2+x+1\right)\sqrt{x^2+x+1}}dx$$

$$\int \frac{x+\frac{1}{2}}{\left(x^2+x+1\right)\sqrt{x^2+x+1}}dx = \left[y=\sqrt{x^2+x+1}\right] = \int \frac{dy}{y^2} = -\frac{1}{y}+C = -\frac{1}{\sqrt{x^2+x+1}}+C,$$

$$\int \frac{1}{\left(x^2+x+1\right)\sqrt{x^2+x+1}}dx = \left[\begin{array}{l} z=\left(\sqrt{x^2+x+1}\right)' = \dfrac{x+\frac{1}{2}}{\sqrt{x^2+x+1}},\, z^2=\dfrac{x^2+x+\frac{1}{4}}{x^2+x+1}, \\ 1-z^2=\dfrac{\frac{3}{4}}{x^2+x+1};\;\; z\sqrt{x^2+x+1}=x+\dfrac{1}{2},\;\sqrt{x^2+x+1}dz+z^2dx=dx, \\ \dfrac{dz}{1-z^2}=\dfrac{dx}{\sqrt{x^2+x+1}} \end{array}\right] =$$

$$= \int \frac{4}{3}\left(1-z^2\right)\frac{dz}{1-z^2} = \frac{4}{3}\int dz = \frac{4}{3}z+C = \frac{2}{3}\frac{2x+1}{\sqrt{x^2+x+1}}+C$$

$$\int \frac{x+1}{\left(x^2+x+1\right)\sqrt{x^2+x+1}}dx = -\frac{1}{\sqrt{x^2+x+1}}+\frac{1}{3}\frac{2x+1}{\sqrt{x^2+x+1}}+C = \frac{2x-2}{3\sqrt{x^2+x+1}}+C$$

1976

$x>0$

$$\int \frac{\left(x^2-1\right)dx}{\left(x^2+1\right)\sqrt{x^4+1}} = \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)\sqrt{x^2+\frac{1}{x^2}}} = \left[y=x+\frac{1}{x}\right] = \int \frac{dy}{y\sqrt{y^2-2}}$$

$$= \left[y=\frac{1}{z}\right] = -\int \frac{dz}{\sqrt{1-2z^2}} = -\frac{1}{\sqrt{2}}\arcsin\sqrt{2}z+C = -\frac{1}{\sqrt{2}}\arcsin\frac{\sqrt{2}x}{x^2+1}+C$$

1985.

$$\int \frac{dx}{\sqrt[3]{1+x^3}} = \left[z^3 = \frac{1}{x^3} + 1,\ x = \left(z^3 - 1\right)^{-1/3} \right] = -\int \left(z^3 - 1\right)^{-4/3} z^2 \frac{1}{\left(1 + \frac{1}{z^3 - 1}\right)^{1/3}} dz =$$

$$= -\int \frac{z}{z^3 - 1} dz = -\frac{1}{3} \int \left(\frac{1}{z - 1} - \frac{z - 1}{z^2 + z + 1} \right) dz = \frac{1}{6} \ln \frac{z^2 + z + 1}{\left(z - 1\right)^2} - \frac{1}{\sqrt{3}} \operatorname{arctg} \frac{2z + 1}{\sqrt{3}} + C$$

Интегрирование тригонометрических выражений

2000.

$$\int \frac{dx}{\cos^3 x} = \int \frac{\cos^2 x + \sin^2 x}{\cos^3 x} dx = \int \frac{dx}{\cos x} + \frac{1}{2} \int \sin x d \frac{1}{\cos^2 x} =$$

$$= \int \frac{dx}{\cos x} + \frac{\sin x}{2\cos^2 x} = \int \frac{\cos x dx}{1 - \sin^2 x} + \frac{\sin x}{2\cos^2 x} = \frac{1}{2} \ln \frac{1 + \sin x}{1 - \sin x} + \frac{\sin x}{2\cos^2 x} + C$$

2025.

$$\int \frac{dx}{2\sin x - \cos x + 5} = \left[u = \operatorname{tg} \frac{x}{2}, du = \frac{dx}{2\cos^2 \frac{x}{2}} = \frac{1}{2}\left(1 + u^2\right) dx \right] =$$

$$= \int \frac{1}{\frac{4u}{1+u^2} - \frac{1-u^2}{1+u^2} + 5} \frac{2du}{1+u^2} = \int \frac{2du}{6u^2 + 4u + 4} = \frac{1}{3} \int \frac{du}{u^2 + \frac{2}{3}u + \frac{2}{3}} =$$

$$= \frac{1}{3} \int \frac{du}{\left(u + \frac{1}{3}\right)^2 + \frac{5}{9}} = \frac{1}{3} \cdot \frac{3}{\sqrt{5}} \operatorname{arctg} \frac{3u + 1}{\sqrt{5}} + C = \frac{1}{\sqrt{5}} \operatorname{arctg} \frac{3\operatorname{tg} \frac{x}{2} + 1}{\sqrt{5}} + C$$

$$\int \frac{\sin^2 x}{\sin^2 x + 2\cos^2 x + 1} dx = \int \frac{\sin^2 x}{2\sin^2 x + 3\cos^2 x} dx = \int \frac{\operatorname{tg}^2 x}{2\operatorname{tg}^2 x + 3} dx = \left[u = \operatorname{tg} x \right] =$$

$$= \int \frac{u^2}{\left(2u^2 + 3\right)\left(u^2 + 1\right)} du = \int \left(\frac{3}{2u^2 + 3} - \frac{1}{u^2 + 1} \right) du = \sqrt{\frac{3}{2}} \operatorname{arctg} \frac{\sqrt{2}u}{\sqrt{3}} - \operatorname{arctg} u + C$$

$$= \sqrt{\frac{3}{2}} \operatorname{arctg} \frac{\sqrt{2}\operatorname{tg} x}{\sqrt{3}} - x + C$$